Implicit matrix representation of curves via quadratic relations

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What are parametric curves?

$$\begin{array}{rcl} \phi := \mathbb{R} & \to & \mathbb{R}^n \\ s & \mapsto & \left(\frac{f_1(s)}{f_0(s)}, \cdots, \frac{f_n(s)}{f_0(s)}\right), \end{array}$$

image of ϕ defines a curve in $\mathbb{R}^n$.

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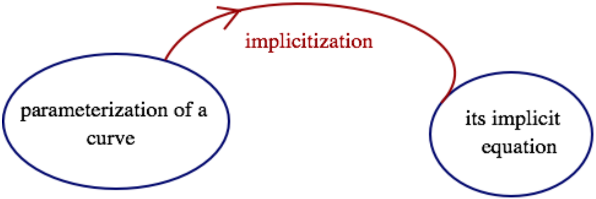
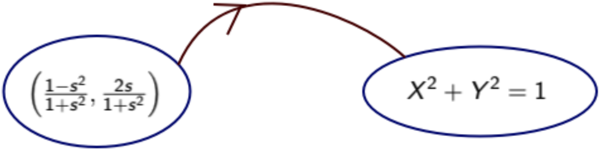
Implicitization



Example

Unit circle in $\mathbb{R}^2$.



What are implicit equations used for ?

Example: (plane curves)

- Is a given point on a given plane curve $\mathcal{C}$?
  $p = (x, y)$: point in $\mathbb{R}^2$,
  $F(T_1, T_2) = 0$: implicit equation of the $\mathcal{C}$.

Question

Is $F(x, y) = 0$?

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Question

Is $F(x, y) = 0$?

Example: (space curves)

- Is a given point $p = (x_1, \cdots, x_n)$ on a given space curve $\mathcal{C}$ in $\mathbb{R}^n$?
  $F_1, \cdots, F_r$ with $F_1 \neq \cdots \neq F_r$ define the $\mathcal{C}$.

Question

Are $F_1(x_1, \cdots, x_n) = 0, \cdots, F_r(x_1, \cdots, x_n) = 0$?

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What are implicit equations used for ?

- Intersection of curves $\mathcal{C}_1$ and $\mathcal{C}_2$, both in $\mathbb{R}^2$:

  $\mathcal{C}_1$ is given by the parameterization

$$\begin{array}{ccc} \mathbb{R} & \to & \mathbb{R}^2 \\ s & \mapsto & \left(\frac{f_1(s)}{f_0(s)}, \frac{f_2(s)}{f_0(s)}\right), \end{array}$$

  $\mathcal{C}_2$ is given by the implicit equation $F(T_1, T_2) = 0$.

Question

- *Is* $F\left(\frac{f_1(s)}{f_0(s)}, \frac{f_2(s)}{f_0(s)}\right) = 0$ *?*
- *If yes, for which s values ?*

What are implicit equations used for ?

- Intersection of the curves $\mathcal{C}_1$ and $\mathcal{C}_2$, both in $\mathbb{R}^n$, $n \geq 2$
  $\mathcal{C}_1$ is given by the parameterization

$$\begin{array}{ccc} \mathbb{R} & \rightarrow & \mathbb{R}^n \\ s & \mapsto & \left(\frac{f_1(s)}{f_0(s)}, \cdots, \frac{f_n(s)}{f_0(s)} \right), \end{array}$$

  $\mathcal{C}_2$ is given by the implicit equations
  $F_1(T_1, \cdots, T_n) = 0, \cdots F_r(T_1, \cdots, T_n) = 0.$

Question

- *Are* $F_1\left(\frac{f_1(s)}{f_0(s)}, \cdots, \frac{f_n(s)}{f_0(s)} \right) = 0, \cdots, F_r\left(\frac{f_1(s)}{f_0(s)}, \cdots, \frac{f_n(s)}{f_0(s)} \right) = 0$ *?*
- *If yes, for which s values ?*

Difficulty

- *Several substitutions,*
- *high degree of polynomials to manipulate.*

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Plane curves

Let $\mathbb{K}$ be a field.
Algebraic parameterization ϕ is defined as follows

$$\begin{array}{rcl} \phi := \mathbb{P} & \to & \mathbb{P}^2 \\ (s:t) & \mapsto & (f_0(s,t) : f_1(s,t) : f_2(s,t))\,, \end{array}$$

and its image defines the curve $\mathcal{C}$.
We assume that the f_i's are of degree d for all $i = 0, 1, 2$.

Plane curves

Implicitization via Sylvester matrix, notation : Syl

T_0, T_1, T_2 : new indeterminates.
$I := (f_0 T_1 - f_1 T_0, f_0 T_2 - f_2 T_0) \subset \mathbb{K}[s, t, T_0, T_1, T_2]$ ideal. I contains implicit equation of the curve $\mathcal{C}$.

Example

$f_0 = s^3 - \frac{1}{2}s^2 t + \frac{5}{9}st^2 - t^3$, $f_1 = 14s^3 + \frac{2}{3}s^2 t - 3st^2 + t^3$,
$f_2 = -\frac{1}{4}s^3 - 12s^2 t - \frac{4}{3}st^2 - 97t^3$. *Then,*
$\mathrm{Syl}(f_0 T_1 - f_1 T_0, f_0 T_2 - f_2 T_0) =$

$$\begin{bmatrix} T_1 - 14T_0 & -\frac{1}{2}T_1 - \frac{2}{3}T_0 & \frac{5}{9}T_1 + 3T_0 & -T_1 - T_0 & 0 & 0 \\ 0 & T_1 - 14T_0 & -\frac{1}{2}T_1 - \frac{2}{3}T_0 & \frac{5}{9}T_1 + 3T_0 & -T_1 - T_0 & 0 \\ 0 & 0 & T_1 - 14T_0 & -\frac{1}{2}T_1 - \frac{2}{3}T_0 & \frac{5}{9}T_1 + 3T_0 & -T_1 - T_0 \\ T_2 + \frac{1}{4}T_0 & -\frac{1}{2}T_2 + 12T_0 & \frac{5}{9}T_2 + \frac{4}{3}T_0 & -T_2 + 97T_0 & 0 & 0 \\ 0 & T_2 + \frac{1}{4}T_0 & -\frac{1}{2}T_2 + 12T_0 & \frac{5}{9}T_2 + \frac{4}{3}T_0 & -T_2 + 97T_0 & 0 \\ 0 & 0 & T_2 + \frac{1}{4}T_0 & -\frac{1}{2}T_2 + 12T_0 & \frac{5}{9}T_2 + \frac{4}{3}T_0 & -T_2 + 97T_0 \end{bmatrix}$$

is a 6×6 *matrix with linear entries in* T_0, T_1, T_2*, and its determinant yields a polynomial of degree* 6 *in* T_0, T_1, T_2.

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Plane curves

Definition

Syzygy module of the parameterization ϕ, denoted by Syz, is

$$\mathrm{Syz}(f_0, f_1, f_2) := \{(p_0, p_1, p_2) \in \mathbb{K}[s,t]^3 : p_0(s,t)f_0(s,t) + p_1(s,t)f_1(s,t) + p_2(s,t)f_2(s,t) = 0\}.$$

p_0, p_1, p_2 are called syzygies of f_0, f_1, f_2. Moreover, $\mathrm{Syz}(f_0, f_1, f_2)$ is a free module of $\mathbb{K}[s,t]$ with 2 generators p and q in $\mathbb{K}[s,t]^3$:

$$p := (p_0, p_1, p_2) \text{ and } q := (q_0, q_1, q_2).$$

Plane curves

Definition

Syzygy module of the parameterization ϕ, denoted by Syz, is

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$$p := (p_0, p_1, p_2) \text{ and } q := (q_0, q_1, q_2).$$

Definition

$\{p, q\}$ are called μ-basis of the parametric curve, if

- *$\{p, q\}$ is a basis of $\mathrm{Syz}(f_0, f_1, f_2)$ and*
- *p, q have the lowest degree among all the basis of $\mathrm{Syz}(f_0, f_1, f_2)$.*

Moreover, $deg(p) = \mu_1$, $deg(q) = \mu_2$ and $d = \mu_1 + \mu_2$. We assume that $\mu_2 \geq \mu_1$.

Implicitization by resultant matrices with respect to p, q

Notation

$\boldsymbol{p} = p_0(s,t)\,T_0 + p_1(s,t)\,T_1 + p_2(s,t)\,T_2$ *and*
$\boldsymbol{q} = q_0(s,t)\,T_0 + q_1(s,t)\,T_1 + q_2(s,t)\,T_2.$

Example

$f_0 = s^3 - \frac{1}{2}s^2t + \frac{5}{9}st^2 - t^3,\ f_1 = 14s^3 + \frac{2}{3}s^2t - 3st^2 + t^3,$
$f_2 = -\frac{1}{4}s^3 - 12s^2t - \frac{4}{3}st^2 - 97t^3.$

$$\begin{bmatrix} p_0 & q_0 \\ p_1 & q_1 \\ p_2 & q_2 \end{bmatrix} = \begin{bmatrix} 2072314393/993502048s + 491833577/124187756t & 1007/84s^2 + 233/168st + 5431/56t^2 \\ -147910417/993502048s - 293063387/1490253072t & -97/112s^2 - 43/504st - 389/56t^2 \\ 1568555/248375512s - 9123809/2128932960t & -23/42s^2 + 97/126st - 15/14t^2 \end{bmatrix}$$

Then, $\mu_1 = 1$ *and* $\mu_2 = 2$. $\mathrm{Syl}(\boldsymbol{p},\boldsymbol{q})$ *is a matrix of* 3×3 *size, with linear entries in* T_0, T_1, T_2*, and its determinant yields a polynomial of degree* 3 *in* T_0, T_1, T_2.

Implicitization by resultant matrices with respect to p, q

Notation

$\boldsymbol{p} = p_0(s,t)T_0 + p_1(s,t)T_1 + p_2(s,t)T_2$ *and*
$\boldsymbol{q} = q_0(s,t)T_0 + q_1(s,t)T_1 + q_2(s,t)T_2.$

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We have $\mu_1 = 1, \mu_2 = 2$.

Definition

Bézout matrix, denoted by $\mathrm{Bez}(\boldsymbol{p},\boldsymbol{q}) = (b_{ij})_{1\le i,j\le \mu_2}$, *is defined to be*

$$\frac{p(\tau,\sigma)q(s,t) - p(s,t)q(\tau,\sigma)}{s\tau - t\sigma} = \sum_{i,j=1} b_{ij} t^{i-1} s^{\mu_2 - i+1} \tau^{j-1} \sigma^{\mu_2 - j+1}.$$

- $\mathrm{Bez}(\boldsymbol{p},\boldsymbol{q})$ is a matrix of 2×2 size, with only quadratic entries in T_0, T_1, T_2, and its determinant yields to a polynomial of degree $2\mu_2$ in T_0, T_1, T_2.

Implicitization by resultant matrices with respect to p, q

Notation

$\boldsymbol{p} = p_0(s,t)T_0 + p_1(s,t)T_1 + p_2(s,t)T_2$ *and*
$\boldsymbol{q} = q_0(s,t)T_0 + q_1(s,t)T_1 + q_2(s,t)T_2$.

- Hybird Bézout matrix, $\mathrm{HBez}(\boldsymbol{p},\boldsymbol{q})$ is composed of the last $\mu_2 - \mu_1$ rows of $\mathrm{Syl}(\boldsymbol{p},\boldsymbol{q})$ in coefficients of q and the first μ_1 rows of $\mathrm{Bez}(\boldsymbol{p},\boldsymbol{q})$. Hence, again for the same example $\mathrm{HBez}(\boldsymbol{p},\boldsymbol{q})$ is a matrix of 2×2 size, with linear and quadratic entries in T_0, T_1, T_2, and its determinant yields a polynomial of degree $d = \mu_2 + \mu_1$ in T_0, T_1, T_2.

$$\mathrm{HBez}(\boldsymbol{p},\boldsymbol{q})^T = \begin{array}{c} \text{last row of } \mathrm{Syl}_s(\boldsymbol{p},\boldsymbol{q}) \\ \text{1st row of } \mathrm{Bez}_s(\boldsymbol{p},\boldsymbol{q}) \end{array} \begin{pmatrix} * & * \\ * & * \end{pmatrix}$$

Hybrid Bézout

$$\boldsymbol{p} = a_0(T_1, T_2)t^{\mu_1} + a_1(T_1, T_2)st^{\mu_1 - 1} + \cdots + a_{\mu_1}(T_1, T_2)s^{\mu_1},$$
$$\boldsymbol{q} = b_0(T_1, T_2)t^{\mu_2} + b_1(T_1, T_2)st^{\mu_2 - 1} + \cdots + b_{\mu_2}(T_1, T_2)s^{\mu_2}.$$

$$\mathrm{Syl}(\boldsymbol{p}, \boldsymbol{q}) = \begin{bmatrix} b_{\mu_2} & b_{\mu_2-1} & & \cdots & b_0 & & \\ & & & & \ddots & & \\ & \ddots & & & & & \\ & & b_{\mu_2} & b_{\mu_2-1} & \cdots & b_0 \\ a_{\mu_1} & a_{\mu_1-1} & \cdots & a_0 & & & \\ & \ddots & & & & & \\ & & & & \ddots & & \\ & a_{\mu_1} & a_{\mu_1-1} & a_{\mu_1-2} & & & \\ & & a_{\mu_1} & a_{\mu_1-1} & \cdots & a_0 \end{bmatrix}$$

$$\mathrm{HBez}(\boldsymbol{p}, \boldsymbol{q})^T = \begin{array}{l} \text{last row of } \mathrm{Syl}(\boldsymbol{p}, \boldsymbol{q}) \\ \text{d-1th row of } \mathrm{Syl}(\boldsymbol{p}, \boldsymbol{q}) \\ \vdots \\ \text{d} - \mu_2 + \mu_1 + \text{1th row of } \mathrm{Syl}(\boldsymbol{p}, \boldsymbol{q}) \\ \text{1st row of } \mathrm{Bez}(\boldsymbol{p}, \boldsymbol{q}) \\ \vdots \\ \mu_1\text{th row of } \mathrm{Bez}(\boldsymbol{p}, \boldsymbol{q}) \end{array} \begin{pmatrix} * & \cdots & * \\ * & \cdots & * \\ & & \\ * & \cdots & * \\ * & \cdots & * \\ * & \cdots & * \\ & & \\ * & \cdots & * \\ * & \cdots & * \end{pmatrix}$$

Remark

If $\mu_2 = \mu_1$, then $\mathrm{HBez}(\boldsymbol{p}, \boldsymbol{q})$ does not have any rows of $\mathrm{Syl}(\boldsymbol{p}, \boldsymbol{q})$, i.e. any rows with linear entries in T_0, T_1, T_2.

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Hybrid Bézout

If $\mu_2 - \mu_1 = 2$, then

$$\mathrm{Syl}(\boldsymbol{p},\boldsymbol{q}) = \begin{bmatrix} b_{\mu_2} & b_{\mu_2-1} & & \cdots & b_0 & \\ & & & & \ddots & \\ & \ddots & & & & \\ & & b_{\mu_2} & b_{\mu_2-1} & \cdots & b_0 \\ a_{\mu_1} & a_{\mu_1-1} & \cdots & a_0 & & \\ & \ddots & & & & \\ & a_{\mu_1} & a_{\mu_1-1} & a_{\mu_1-2} & \cdots & 0 \\ & 0 & a_{\mu_1} & a_{\mu_1-1} & \cdots & a_0 \end{bmatrix}$$

The red block in $\mathrm{Syl}(\boldsymbol{p},\boldsymbol{q})$ corresponds to the monomial basis $\{s^{\mu_1+1}t^{\mu_2-1}, s^{\mu_1}t^{\mu_2}, \cdots, st^{d-1}, t^d\}$ as columns.

$$\mathrm{HBez}(\boldsymbol{p},\boldsymbol{q})^T = \begin{array}{l} \text{last row of } \mathrm{Syl}(\boldsymbol{p},\boldsymbol{q}) \\ \text{d-1th row of } \mathrm{Syl}(\boldsymbol{p},\boldsymbol{q}) \\ \vdots \\ d - \mu_2 + \mu_1 + 1\text{th row of } \mathrm{Syl}(\boldsymbol{p},\boldsymbol{q}) \\ \text{1st row of } \mathrm{Bez}(\boldsymbol{p},\boldsymbol{q}) \\ \vdots \\ \mu_1\text{th row of } \mathrm{Bez}(\boldsymbol{p},\boldsymbol{q}) \end{array} \begin{pmatrix} * & \cdots & * \\ * & \cdots & * \\ & & \\ * & \cdots & * \\ * & \cdots & * \\ * & \cdots & * \\ & & \\ * & \cdots & * \\ * & \cdots & * \end{pmatrix}$$

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Another interpretation of the quadratic part of HBez

Sylvester form of the μ-basis p, q

$\alpha := (\alpha_1, \alpha_2) \in \mathbb{Z}_{\geq 0}$, such that $|\alpha| := \alpha_1 + \alpha_2 \leq \mu_1 - 1$.
$\boldsymbol{p}$ and $\boldsymbol{q}$ can be decomposed as

$$\boldsymbol{p} = s^{\alpha_1+1} h_{1,1} + t^{\alpha_2+1} h_{1,2},$$
$$\boldsymbol{q} = s^{\alpha_1+1} h_{2,1} + t^{\alpha_2+1} h_{2,2},$$

where $h_{i,j}(s, t; x_0, x_1, x_2)$ are homogeneous polynomials of degree $\mu_i - \alpha_j - 1$ with respect to the variables s, t and linear in T_0, T_1, T_2.

Definition

The polynomial

$$\mathrm{Syl}_\alpha(\boldsymbol{p}, \boldsymbol{q}) := \det \begin{pmatrix} h_{1,1} & h_{1,2} \\ h_{2,1} & h_{2,2} \end{pmatrix}$$

is called Sylvester form of the μ-basis.

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where $h_{i,j}(s,t;x_0,x_1,x_2)$ are homogeneous polynomials of degree $\mu_i - \alpha_j - 1$ with respect to the variables s, t and linear in T_0, T_1, T_2.

Definition

The polynomial

$$\mathrm{Syl}_\alpha(\boldsymbol{p}, \boldsymbol{q}) := det \begin{pmatrix} h_{1,1} & h_{1,2} \\ h_{2,1} & h_{2,2} \end{pmatrix}$$

is called Sylvester form of the μ-basis.

Theorem

Let ν be an integer such that $\mu_2 - 1 \leq \nu \leq d - 2$. Then the set of $d - 1 - \nu$ Sylvester forms

$$\{\mathrm{Syl}_\alpha(\boldsymbol{p}, \boldsymbol{q})\}_{|\alpha|=d-2-\nu} = \left\{\mathrm{Syl}_{(d-2-\nu,0)}(\boldsymbol{p}, \boldsymbol{q}), \ldots, \mathrm{Syl}_{(0,d-2-\nu)}(\boldsymbol{p}, \boldsymbol{q})\right\}$$

form a basis of the quadratic part of $\mathrm{HBez}(\boldsymbol{p}, \boldsymbol{q})$.

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Summary

Assume $\deg(f_i) = d, \forall i = 0, 1, 2$ and $\mu_2 \geq \mu_1$.
For a general plane curve of degree d

$$\mu_2 = \left\lceil \frac{d}{2} \right\rceil.$$

size of the matrix	type of resultant matrix	degree of determinant
$(2d \times 2d)$	$\mathrm{Syl}(f_0 T_1 - f_1 T_0, f_0 T_2 - f_2 T_0)$	$2d$,
$(d \times d)$	$\mathrm{Syl}(\boldsymbol{p}, \boldsymbol{q})$	d,
$(\mu_2 \times \mu_2)$	$\mathrm{HBez}(\boldsymbol{p}, \boldsymbol{q})$	d.

- μ-basis serves to decrease the size of Syl matrix to its half size,
- $\mathrm{HBez}(\boldsymbol{p}, \boldsymbol{q})$ has half size of $\mathrm{Syl}(\boldsymbol{p}, \boldsymbol{q})$.

Existing method : Syzygy based matrix M

There exist already a method which generalizes Syl of μ-basis into higher dimensions.
Let $\mathbb{K}$ be a field.

$$\begin{array}{rcl} \phi := \mathbb{P}^1 & \to & \mathbb{P}^n \\ (s:t) & \mapsto & (f_0(s,t) : f_1(s,t) : \cdots : f_n(s,t)) \,. \end{array}$$

Definition

Syzygy module of the parameterization ϕ, denoted by Syz, is

$$\mathrm{Syz}(f_0, \cdots, f_n) := \{(g_0, \cdots, g_n) \in \mathbb{K}[s,t]^{n+1} : \sum_{i=0}^n g_i f_i = 0\}.$$

$g_0, \cdots, g_n$ are called syzygies of $f_0, \cdots, f_n$. Moreover, $\mathrm{Syz}(f_0, \cdots, f_n)$ is a free module of $\mathbb{K}[s,t]$ with n generators $p_1, \cdots, p_n$ in $\mathbb{K}[s,t]^{n+1}$: $p_i := (p_{i_0}, \cdots, p_{i_n}), \forall i = 1, \cdots n$.

Definition

$\{p_1, \cdots, p_n\}$ are called μ-basis of the parametric curve, if

- *$\{p_1, \cdots, p_n\}$ is a basis of $\mathrm{Syz}(f_0, \cdots, f_n)$ and*
- *$\{p_1, \cdots, p_n\}$ have the lowest degree among all the basis of $\mathrm{Syz}(f_0, \cdots, f_n)$.*

Moreover, $deg(p_i) = \mu_i, \forall i = 1, \cdots, n$, and $\sum_{i=1}^n \mu_i = d$. We assume that $\mu_n \geq \cdots \geq \mu_1$.

Existing method : Syzygy based matrix M

There exist already a method which generalizes Syl of μ-basis into higher dimensions. Let $\mathbb{K}$ be a field.

$$\begin{array}{rcl} \phi := \mathbb{P} & \to & \mathbb{P}^n \\ (s:t) & \mapsto & (f_0(s,t) : f_1(s,t) : \cdots : f_n(s,t)) \,. \end{array}$$

$T_0, \cdots, T_n$: new indeterminates.

Assumption

$\mu_n \geq \cdots \geq \mu_1$.

M is computed at degree $\mu_n + \mu_{n-1} - 1$, i.e. its rows are in monomials basis $\{t^{\mu_n+\mu_{n-1}-1}, st^{\mu_n+\mu_{n-1}-2}, \dots, s^{\mu_n+\mu_{n-1}-1}\}$, so it has $\mu_n + \mu_{n-1}$ rows with linear entries in $T_0, \cdots, T_n$.

What is M ?

M considers moving lines.

What is a moving line?

A moving line L is

$$L = A_0(s,t)T_0 + A_1(s,t)T_1 + \cdots + A_n(s,t)T_n.$$

We say that L follows the surface if

$$\sum_{i=1}^{n} A_i(s,t)\phi_i(s,t) \equiv 0.$$

L is of degree 1 in $T_0, \cdots, T_n$.

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What is M ?

M is constructed by the coefficients of the family of moving lines of degree $\mu_n + \mu_{n-1} - 1$ over s, t

$$M_{\mu_n+\mu_{n-1}-1} = \begin{bmatrix} | & | & | \\ | & | & | \\ L_1 & | & L_r \\ | & | & | \\ | & | & | \end{bmatrix}$$

such that

$$\left(s^{\mu_n+\mu_{n-1}-1}, s^{\mu_n+\mu_{n-1}-2}t, \cdots, t^{\mu_n+\mu_1-1}\right) M_{\mu_n+\mu_{n-1}-1} = [L_1, \cdots, L_r].$$

The L_i's are the moving lines following the parametrization of the given curve.

What is M ?

- M considers only linear relations, (as Syl),
- In $\mathbb{P}^2$, M is computed at degree $\mu_2 + \mu_1 - 1 = d - 1$, so it has d rows with linear entries in T_0, T_1, T_2.

size of the matrix	type of resultant matrix	degree of determinant
$(d \times d)$	$\mathrm{Syl}(\boldsymbol{p}, \boldsymbol{q})$	d,
$(d \times d)$	$M_{\mu_2+\mu_1-1}$	d,
$(\mu_2 \times \mu_2)$	$\mathrm{HBez}(\boldsymbol{p}, \boldsymbol{q})$	d.

- M works for higher dimensions, i.e. spaces curves in $\mathbb{P}^n$, $n \geq 3$,
- It is written in a monomial or Bézier basis of degree $\mu_n + \mu_{n-1} - 1$,
- The rank of M drops for the points on the curve $\mathcal{C}$.

Our new method, notation : QM

Why a new method ?

- QM generalizes Hybrid Bézout to the higher dimensions,

$$\mathrm{HBez} \in \mathbb{P}^2 \rightsquigarrow QM \in \mathbb{P}^n, n \geq 3,$$

- The rows of QM are in monomial basis of degree $\mu_n - 1$,

number of rows	type of matrix
$\mu_n + \mu_{n-1}$	$M_{\mu_n+\mu_{n-1}-1}$,
μ_n	QM_{μ_n} .

  We recall that for a general curve $\mu_i = \left\lfloor \frac{d}{n} \right\rfloor$, for $i = 1, \cdots n-1$, and $\mu_n = \left\lceil \frac{d}{n} \right\rceil$, hence QM has almost the half rows of M.

- The rank of QM drops for the points $(x_0, \cdots, x_n)$ on the curve $\mathcal{C} \in \mathbb{R}^n$.

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Our new method *QM*

QM considers both moving lines and moving quadrics.

What is a moving quadric?

A moving quadric L is

$$Q = A_{00}(s,t)\,T_0^2 + A_{01}(s,t)\,T_0 T_1 + \cdots + A_{nn}(s,t)\,T_n^2.$$

We say that Q follows the surface if

$$\sum_{1 \le i \le j \le n} A_{ij}(s,t)\phi_i(s,t)\phi_j(s,t) \equiv 0.$$

Q is of degree 2 in $T_0, \cdots T_n$.

Our new method *QM*

Remark

Let L be a moving line following the parameterization ϕ. Then,

$$T_i L = T_i (A_0(s,t) T_0 + \cdots + A_n(s,t) T_n), \forall i = 0, \cdots, n$$

is a moving quadric following the parameterization ϕ.

Hence, we consider the subvector space of moving quadrics which are not coming from moving lines.

Our new method QM

What is QM?

QM is constructed by the coefficients of the family of moving quadrics which are not coming from moving lines, and moving lines of degree $\mu_n - 1$ over s, where $\mu_n \geq \cdots \geq \mu_1$.

$$QM_{\mu_n-1} = \begin{bmatrix} | & | & | & | & | & | \\ | & | & | & | & | & | \\ L_1 & | & L_r & Q_1 & | & Q_k \\ | & | & | & | & | & | \\ | & | & | & | & | & | \end{bmatrix}$$

such that

$$\left(s^{\mu_n-1}, s^{\mu_n-2}t, \cdots, t^{\mu_n-1}\right) QM_{\mu_n-1} = [L_1, \cdots, L_r, Q_1, \cdots Q_k].$$

The L_i's are the moving lines and the Q_j's are the moving quadrics following the parametrization of the given curve.

Main result

We have a compact implicit matrix QM of a parametric curve $\mathcal{C}$ in $\mathbb{P}^n$ with linear and quadratic entires in $T_0, \cdots, T_n$, in monomial basis of degree $\mu_n - 1$, such that on the points $(x_0, \cdots, x_n) \in \mathcal{C}$, the rank of $QM(x_1, \cdots, x_n)$ drops.

Comparisons

For a general degree 8 parametric curve, having μ-basis of degree $(2, 3, 3)$, we have following number of linear and quadratic relations according to monomial basis in chosen degree.
BLUE : M,
RED : QM.

degree of monomial basis	linear relations	quadratic relations	size of QM
1	0	2	2×2
2	1	7	3×8
3	4	4	4×8
4	7	1	5×8
5	10	0	6×10
6	13	0	7×13
7	16	0	8×16

- M appears from the degree 5 which is $\mu_n + \mu_{n-1} - 1 = 3 + 3 - 1 = 5$.
- QM appears from the degree 2 which is $\mu_n - 1 = 3 - 1 = 2$.

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Is point on the curve?

- Check the drop of rank of QM evaluated at given point.

d	μ_i's	M size	Mrep ms.	rank ms.	QM size	QM ms	rank ms
5	(2,3)	5x5	9	4.13	3x3	14	1.69
5	(1,2,2)	4x7	7	4.3	2x5	18	1.62
9	(3,3,3)	6x9	12	8.95	3x9	38	4.64
9	(1,4,4)	8x15	18	18.17	4x9	50	5.18
10	(5,5)	10x10	15	17.83	5x5	28	4.71
15	(1,7,7)	14x27	77	71.3	7x15	282	13.51

- QM has half number of M,
- Computation of QM takes more time than the computation of M, however for instance computation of drop of rank is faster than M.

Theorem

Drop of rank of QM at a given point on $\mathcal{C}$ gives multiplicty of the point.

Thanks!