Factoring polynomials over discrete valuation rings

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Journées Nationales de Calcul Formel

CIRM, Luminy

One example

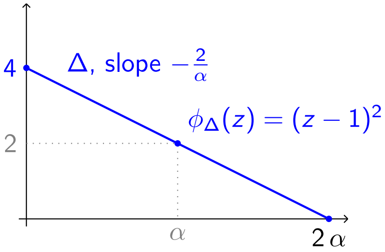
$$F = (y^\alpha - x^2)^2 + x^\alpha \in \mathbb{A}[y] \text{ with } \mathbb{A} = \mathbb{C}[[x]]$$

- $d = \deg(F) = 2\,\alpha$,
- $\delta = \upsilon_x(\mathrm{Disc}(F)) = 2\,\alpha^2 - 4\,\alpha + 4$.
- Assume $\alpha > 4$ odd.

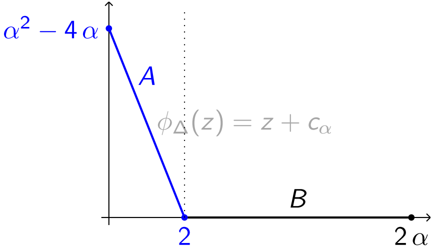
Is F irreducible in $\mathbb{C}[[x]][y]$?

Using the Newton-Puiseux algorithm.

$$F = (y^\alpha - x^2)^2 + x^\alpha$$



1. $G \leftarrow F(x^\alpha, x^2(y+1))/x^{4\,\alpha}$,
2. Hensel: $G = A \cdot B$,
3. Recursive call with A

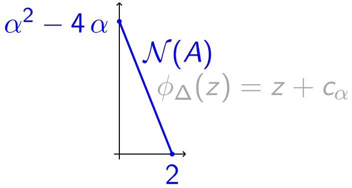


Polynomial	size
F	$\Theta(\alpha^2) = \Theta(\delta)$
G	$\Theta(\alpha^3) = \Theta(d\,\delta)$
A	$\Theta(\alpha^2) = \Theta(\delta)$

Answer: Yes complexity: $\Theta(d\,\delta)$ Answer in $\mathcal{O}(\delta)$?

Another way ?

$$F = (y^\alpha - x^2)^2 + x^\alpha$$



- Writing $\psi = y^\alpha - x^2$, we have $F = \psi^2 + x^\alpha$,
  - Can we "guess" the second Newton polygon from $\psi^2 + x^\alpha$?
  - Can we "read" ϕ_Δ ?

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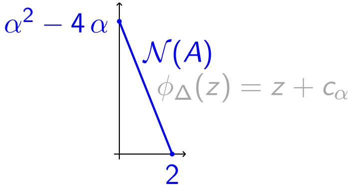

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- Key ingredients:
  - $\psi = \sqrt[2]{F}$ is an approximate root of F,
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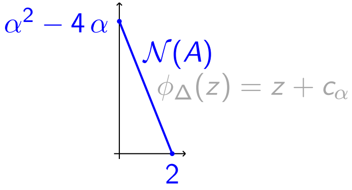

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- Questions:
  - Why x^α corresponds to $\alpha^2 - 4\,\alpha$?
  - How to recover the correct characteristic polynomial ?

This talk

Context:

- $\mathbb{A}$ a discrete valuation ring (e.g. $\mathbb{K}((x))$, $\mathbb{Q}_p$),
- $v_{\mathbb{A}}$ valuation over $\mathbb{A}$ (e.g. v_x, v_p),
- $F \in \mathbb{A}[y]$ (monic).

Objective(s):

1. Irreducibility test in $\mathbb{A}[y]$,
2. Factorisation of F in $\mathbb{A}[y]$.
3. Case $\mathbb{A} = \mathbb{K}[[x]]$: Puiseux series of F ?

Notations: $d = \deg(F)$; $\delta = v_{\mathbb{A}}(\mathrm{Disc}(F))$

Approximate root of $F \in \mathbb{A}[y]$ monic [Ab10]

- Hyp: $\mathrm{char}(\mathbb{A})$ does not divide d,
- Let $N \in \mathbb{N}$ dividing d,

Proposition

There is an unique monic $\psi \in \mathbb{A}[y]$ such that:

- $\deg(\psi) = d/N$,
- $\deg(F - \psi^N) < d - d/N$,

$\rightsquigarrow$ $\psi = \sqrt[N]{F}$ is the N-th approximate root of F.

Example: $\psi = \sqrt[d]{F} = y + \dfrac{a_{d-1}}{d}$ is the d-th approximate root of F.

Valuations on $\mathbb{A}[y]$

- Gauss valuation:
  - $F = \sum_i a_i\, y^i$,
  - $v_0(F) = \min_i v_{\mathbb{A}}(a_i)$.
- Extended valuation: given $\psi \in \mathbb{A}[y]$ monic, $\frac{m}{q} \in \mathbb{Q}$:
  - $v_\psi = (v_0, \psi, \frac{m}{q})$ *extends* v_0.

    Defined by $v_\psi(\psi) = m\,q$, $v_\psi(y) = m$ and $v_\psi(x) = q$,
  - Expand $F = \sum_i a_i(y)\,\psi^i$ with $\deg(a_i) < \deg(\psi)$,
  - Generalised Newton polygon:

    $\mathcal{N}_\psi(F)$ is the lower convex hull of $(i, v_\psi(a_i\,\psi^i) - v_\psi(F))_i$.

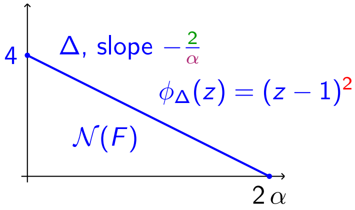
Improving the irreducibility test

generalisation of the work of Abhyankhar to $\mathbb{A}[y]$.

link with the Newton–Puiseux algorithm for $\mathbb{A} = \mathbb{K}((x))$

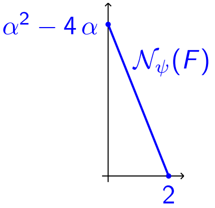
We get the second Newton polygon !

$$F = (y^\alpha - x^2)^2 + x^\alpha$$



With $m = 2$, $q = \alpha$, $\psi = \sqrt[2]{F}$, we get:

- $F = \psi^2 + x^\alpha$.
- $v_\psi(F) = 4\,\alpha$
- $v_\psi(\psi^2) - v_\psi(F) = 0$,
- $v_\psi(x^\alpha) - v_\psi(F) = \alpha^2 - 4\,\alpha$.



Reminder: $v_\psi(x) = \alpha \quad v_\psi(y) = 2 \quad v_\psi(\psi) = 2\,\alpha$

Complexity ?

- Computing $\sqrt[N]{F}$: $\mathcal{O}(\mathsf{M}(d)) = \tilde{\mathcal{O}}(d)$ op in $\mathbb{A}$.
  - $F_\infty = y^d\, F(1/y)$ the reciprocal polynomial of F,
  - $F_\infty(0) = 1 \rightsquigarrow \exists!\, \phi \in \mathbb{A}[[y]]$ s.t. $\phi(0) = 1$ and $\phi^N = F_\infty$,
  - ϕ is the root of $Z^N - F_\infty = 0 \rightsquigarrow$ Newton iteration !
  - ψ is the reciprocal polynomial of $\lceil \phi \rceil^{\frac{d}{N}}$

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  - Recursive call on A and B.
- Truncation: $n = 2\,\delta/d$.

Total cost: $\delta\,\mathrm{plog}(d)$!

Miscellaneous

- More than one Newton–Puiseux recursive call ?
  - Compute successive approximate roots $\psi_0, \cdots, \psi_k$ $\qquad \psi_{-1} = x$
  - Recursive augmented valuations $\upsilon_k = (\upsilon_{k-1}, \psi_k, \frac{m_k}{q_k})$:

$$\begin{cases} \upsilon_k(\psi_i) = q_k \upsilon_{k-1}(\psi_i) & -1 \leq i < k-1 \\ \upsilon_k(\psi_{k-1}) = q_k \upsilon_{k-1}(\psi_{k-1}) + m_k \\ \upsilon_k(\psi_k) = q_k \upsilon_k(\psi_{k-1}) \end{cases}$$

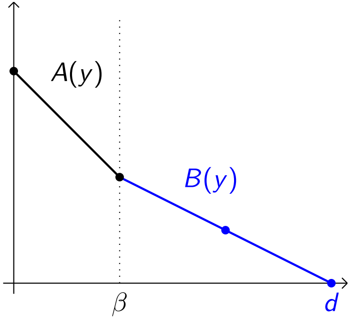
  - $\mathcal{N}_k(F)$ via generalised $(\psi_0, \cdots, \psi_k)$-adic expansions
- Compute the characteristic polynomials ?
  - The coefficients of the ψ-adic expansions must be *corrected*,
  - Compute some $\lambda_k(\psi_i) \in \mathbb{K}_k$ (tower of fields).
- Make a single (univariate) irreducibility test ?
  - Rely on dynamic evaluation.

Hensel–Newton algorithm and extended valuations

Slope factorisation [CaRoVa16]

$$F(y) = \sum_{i=0}^{d} a_i\, y^i$$

- β a "break" of $\mathcal{N}(F)$,
- $A_0 = \sum_{i=0}^{\beta} a_i y^i,\ V_0 = 1,$



Newton iteration:

$$\begin{aligned} A_{k+1} &= A_k + (V_k\, F \% A_k) \\ B_{k+1} &= F \mathbin{/\!/} A_{k+1} \\ V_{k+1} &= (2\,V_k - V_k^2\, B_{k+1}) \% A_{k+1} \end{aligned}$$

Factorisation up to precision $n \rightsquigarrow \tilde{\mathcal{O}}(n\, d)$

Hensel lemma works with extended valuations

Lemma

Assume $B = \psi^b + \cdots$ *and* $v(B) = b\,v(\psi)$. *Then*

- $v(A\%B) \geq v(A)$,
- $v(A \mathbin{/\!\!/} B) \geq v(A) - v(B)$.

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Theorem

Assume

- $v(F - G\,H) \geq v(F) + n$ *and* $v(S\,G + T\,H - 1) \geq n$.

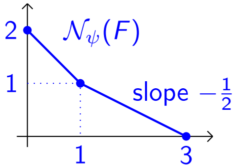
Then $\tilde{G}, \tilde{H}, \tilde{S}, \tilde{T} = \texttt{HenselStep}(F, G, H, S, T)$ *satisfies:*

- $v(F - \tilde{G}\,\tilde{H}) \geq v(F) + 2\,n$,
- $v(\tilde{S}\,\tilde{G} + \tilde{T}\,\tilde{H} - 1) \geq 2\,n$.

Good initialisation ?

$$F = \psi^3 + y^2 x^3 \psi + x^6 y \text{ with } \psi = y^3 - x^2$$

- $v_\psi(x) = 3$, $v_\psi(y) = 2$, $v_\psi(\psi) = 6$.
- Extend v_ψ with the lower edge:
  $v(x) = 6$, $v(y) = 4$, $v(\psi) = 13$



- $G_0 = \overbrace{\psi^2 + y^2 x^3}^{26}$, $H_0 = \overbrace{\psi}^{13} \Longrightarrow \overbrace{F}^{39} - G_0 H_0 = \overbrace{x^6 y}^{40}$

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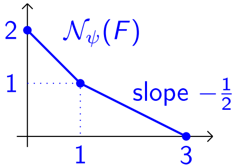

- $G_0 = \overbrace{\psi^2 + y^2 x^3}^{26}$, $H_0 = \overbrace{\psi}^{13} \implies \overbrace{F}^{39} - G_0 H_0 = \overbrace{x^6 y}^{40}$
- With $s_0 = 1$ and $t_0 = -T$, we have $s_0 (T^2 + 1) + t_0 T = 1$,
- $S_0 = \underbrace{x^{-5} y}_{-26}$, $T_0 = \underbrace{-x^{-5} y \psi}_{-13} \implies S_0 G_0 + T_0 H_0 - 1 = \underbrace{x^{-2}\psi}_{1}$

State of the art (sketch)

- Abhyankar-Moh [Ab06]: approximate roots,
- Mac Lane, Abhyankar [Ma36², Ab90, Ru14]: extended valuations,
- Montes et al [Mo99,GuMoNa11&12,BaNaSt13,GuNaPa12] $\mathcal{O}(d^2 + d\delta^2)$,
- Caruso et al [CaRoVa16]: slope factorisation,

Case $\mathbb{A} = \mathbb{K}[[x]]$:

- Sasaki et al [KaSa99,AlAtMa17]: Extended Hensel Construction at least $\mathcal{O}(d^2\,(\delta + d^2))$,
- Puiseux [PoRy15,PoWe]: Newton–Puiseux algorithm $\mathcal{O}(d\,\delta)$.

Conclusion

- Irreducibility test in $\mathbb{A}[y]$ in $\mathcal{O}(\delta)$, $\leftarrow$ improved by a factor d !
- "direct" factorisation in $\mathbb{A}[y]$: $\mathcal{O}(\rho\, n\, d)$, $\leftarrow$ was $\mathcal{O}(n\, d^2)$
- Sage prototype,
- "Bivariate" computations above the *residue field* of $\mathbb{A}$ (no field extension).
- Puiseux series ?
  - $N_1 = d/2$: $\psi_1 = \psi_0^2 + X^{m_1} S_1(X)^2$

  $\rightsquigarrow$ $S_1(X)$ is an approximate root ($\rightsquigarrow$ Newton iteration !)
  - $q_1 > 2$? Solving some linear system ?

Example: if $S_1(x) = x^{\frac{1}{3}}\, P_1(x) + x^{\frac{2}{3}}\, P_2(x)$,

$$\psi_1 = \psi_0^3 - 3\, x\, P_1\, P_2\, \psi_0 - x\, P_1^3 - x^2\, P_2^3$$

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