From moments to sparse representations, a geometric, algebraic and algorithmic viewpoint

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Sparse representation problems

Sparse representation of sequences

Given a sequence of values

$$\sigma_0, \sigma_1, \ldots, \sigma_s \in \mathbb{C},$$

find/guess the values of σ_n for all $n \in \mathbb{N}$.

☞ Find $r \in \mathbb{N}, \omega_i, \xi_i \in \mathbb{C}$ such that $\sigma_n = \sum_1^{\mathbf{r}} \omega_{\mathbf{i}} \, \xi_{\mathbf{i}}{}^n$, for all $n \in \mathbb{N}$.

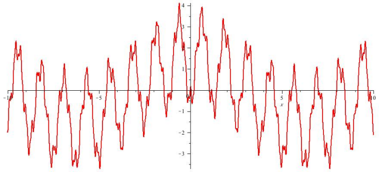
Example: 0, 1, 1, 2, 3, 5, 8, 13, ...

Solution:

- Find a recurrence relation valid for the first terms: $\sigma_{k+2} - \sigma_{k+1} - \sigma_k = 0$.
- Find the roots $\xi_1 = \frac{1+\sqrt{5}}{2}$, $\xi_2 = \frac{1-\sqrt{5}}{2}$ (golden numbers) of the characteristic polynomial: $x^2 - x - 1 = 0$.
- Deduce $\sigma_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$.

Sparse representation of signals

Given a function or signal $f(t)$:



decompose it as

$$f(t) = \sum_{i=1}^{r'} (a_i cos(\mu_i t) + b_i sin(\mu_i t)) e^{\nu_i t} = \sum_{i=1}^{\mathbf{r}} \omega_{\mathbf{i}} e^{\zeta_{\mathbf{i}} t}$$

Prony's method (1795)

For the signal $f(t) = \sum_{i=1}^{\mathbf{r}} \omega_{\mathbf{i}} e^{\zeta_{\mathbf{i}} t}$, $(\omega_i, \zeta_i \in \mathbb{C})$,

- Evaluate f at $2\,r$ regularly spaced points: $\sigma_0 := f(0), \sigma_1 := f(1), \ldots$
- Compute a non-zero element $\mathbf{p} = [\mathbf{p_0}, \ldots, \mathbf{p_r}]$ in the kernel:

$$\begin{bmatrix} \sigma_0 & \sigma_1 & \ldots & \sigma_r \\ \sigma_1 & & & \sigma_{r+1} \\ \vdots & & & \vdots \\ \sigma_{r-1} & \ldots & \sigma_{2r-1} & \sigma_{2r-1} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_r \end{bmatrix} = 0$$

- Compute the roots $\xi_{\mathbf{1}} = \mathbf{e}^{\zeta_1}, \ldots, \xi_{\mathbf{r}} = \mathbf{e}^{\zeta_r}$ of $p(x) := \sum_{i=0}^{r} p_i x^i$.
- Solve the system

$$\begin{bmatrix} 1 & \ldots & \ldots & 1 \\ \xi_1 & & & \xi_r \\ \vdots & & & \vdots \\ \xi_1^{r-1} & \ldots & \ldots & \xi_r^{r-1} \end{bmatrix} \begin{bmatrix} \omega_{\mathbf{1}} \\ \omega_{\mathbf{2}} \\ \vdots \\ \omega_{\mathbf{r}} \end{bmatrix} = \begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \vdots \\ \sigma_{r-1} \end{bmatrix}.$$

Symmetric tensor decomposition and Waring problem (1770)

Symmetric tensor decomposition problem:

Given a homogeneous polynomial ψ of degree d in the variables $\bar{\mathbf{x}} = (x_0, x_1, \ldots, x_n)$ with coefficients $\in \mathbb{K}$:

$$\psi(\bar{\mathbf{x}}) = \sum_{|\alpha|=d} \sigma_\alpha \binom{d}{\alpha} \bar{\mathbf{x}}^\alpha,$$

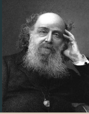
find a minimal decomposition of ψ of the form

$$\psi(\bar{\mathbf{x}}) = \sum_{i=1}^{r} \omega_i (\xi_{i,0} x_0 + \xi_{i,1} x_1 + \cdots + \xi_{i,n} x_n)^d$$

with $\xi_i = (\xi_{i,0}, \xi_{i,1}, \ldots, \xi_{i,n}) \in \overline{\mathbb{K}}^{n+1}$ spanning disctint lines, $\omega_i \in \overline{\mathbb{K}}$.

The minimal r in such a decomposition is called the **rank** of ψ.

Sylvester approach (1851)



Theorem

The binary form $\psi(x_0, x_1) = \sum_{i=0}^{d} \sigma_i \binom{d}{i} x_0^{d-i} x_1^i$ *can be decomposed as a sum of* r *distinct powers of linear forms*

$$\psi = \sum_{k=1}^{r} \omega_k (\alpha_{\mathbf{k}} x_0 + \beta_{\mathbf{k}} x_1)^d$$

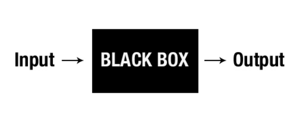
iff there exists a polynomial $p(x_0, x_1) := p_0 x_0^r + p_1 x_0^{r-1} x_1 + \cdots + p_r x_1^r$ *s.t.*

$$\begin{bmatrix} \sigma_0 & \sigma_1 & \dots & \sigma_r \\ \sigma_1 & & & \sigma_{r+1} \\ \vdots & & & \vdots \\ \sigma_{d-r} & \dots & \sigma_{d-1} & \sigma_d \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_r \end{bmatrix} = 0$$

and of the form $p = c \prod_{k=1}^{r} (\beta_{\mathbf{k}} x_0 - \alpha_{\mathbf{k}} x_1)$ *with* $(\alpha_k : \beta_k)$ *distinct.*

Sparse interpolation

Given a black-box polynomial function $f(x)$



find what are the terms inside from output values.

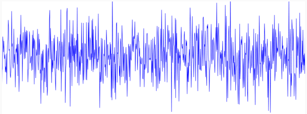
☞ Find $r \in \mathbb{N}, \omega_i \in \mathbb{C}, \alpha_i \in \mathbb{N}$ such that $f(x) = \sum_{i=1}^{r} \omega_i \, x^{\alpha_i}$.

- Choose $\varphi \in \mathbb{C}$
- Compute the sequence of terms $\sigma_0 = f(1), \ldots, \sigma_{2r-1} = f(\varphi^{2r-1})$;
- Construct the matrix $H = [\sigma_{i+j}]$ and its kernel $p = [p_0, \ldots, p_r]$ s.t.

$$\begin{bmatrix} \sigma_0 & \sigma_1 & \ldots & \sigma_r \\ \sigma_1 & & & \sigma_{r+1} \\ \vdots & & & \vdots \\ \sigma_{r-1} & \ldots & \sigma_{2r-1} & \sigma_{2r-1} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_r \end{bmatrix} = 0$$

- Compute the roots $\xi_\mathbf{1} = \varphi^{\alpha_\mathbf{1}}, \ldots, \xi_\mathbf{r} = \varphi^{\alpha_\mathbf{r}}$ of $p(x) := \sum_{i=0}^{r} p_i x^i$ and deduce the exponents $\alpha_\mathbf{i} = \log_\varphi(\xi_\mathbf{i})$.
- Deduce the weights $W = [\omega_i]$ by solving $V_\Xi W = [\sigma_0, \ldots, \sigma_{r-1}]$ where V_Ξ is the Vandermonde system of the roots $\xi_1, \ldots, \xi_r$.

Decoding



An algebraic code:

$$E = \{c(f) = [f(\xi_1), \ldots, f(\xi_m)] \mid f \in \mathbb{K}[x]; \deg(f) \leq d\}.$$

Encoding messages using the dual code:

$$C = E^{\perp} = \{\mathbf{c} \mid \mathbf{c} \cdot [f(\xi_1), \ldots, f(\xi_m)] = 0 \;\forall f \in V = \langle \mathbf{x}^{\mathbf{a}} \rangle \subset \mathbb{F}[\mathbf{x}]\}$$

Message received: $r = m + e$ for $m \in C$ where $e = [\omega_1, \ldots, \omega_m]$ is an error with $\omega_j \neq 0$ for $j = i_1, \ldots, i_r$ and $\omega_j = 0$ otherwise.

☞ **Find the error e.**

Berlekamp-Massey method (1969)

- Compute the syndrome $\sigma_k = c(x^k) \cdot r = c(x^k) \cdot e = \sum_{j=1}^{r} \omega_{i_j} \xi_{i_j}^k$.
- Compute the matrix

$$\begin{bmatrix} \sigma_0 & \sigma_1 & \dots & \sigma_r \\ \sigma_1 & & & \sigma_{r+1} \\ \vdots & & & \vdots \\ \sigma_{r-1} & \dots & \sigma_{2r-1} & \sigma_{2r-1} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_r \end{bmatrix} = 0$$

  and its kernel $p = [p_0, \dots, p_r]$.
- Compute the roots of the error locator polynomial $p(x) = \sum_{i=0}^{r} p_i\, x^i = p_r \prod_{j=1}^{r} (x - \xi_{i_j})$.
- Deduce the errors ω_{i_j}.

Simultaneous decomposition

Simultaneous decomposition problem

Given symmetric tensors $\psi_1, \ldots, \psi_m$ of order $d_1, \ldots, d_m$, find a simultaneous decomposition of the form

$$\psi_l = \sum_{i=1}^{r} \omega_{l,i} (\xi_{i,0} x_0 + \xi_{i,1} x_1 + \cdots + \xi_{i,n} x_n)^{d_l}$$

where $\xi_i = (\xi_{i,0}, \ldots, \xi_{i,n})$ span distinct lines in $\overline{\mathbb{K}}^{n+1}$ and $\omega_{l,i} \in \overline{\mathbb{K}}$ for $l = 1, \ldots, m$.

Proposition (One dimensional decomposition)

Let $\psi_l = \sum_{i=0}^{d_l} \sigma_{1,i} \binom{d_l}{i} x_0^{d_l - i} x_1^i \in \mathbb{K}[x_0, x_1]_{d_l}$ *for* $l = 1, \ldots, m$.
If there exists a polynomial $p(x_0, x_1) := p_0 x_0^r + p_1 x_0^{r-1} x_1 + \cdots + p_r x_1^r$ *s.t.*

$$\left[\begin{array}{cccc} \sigma_{1,0} & \sigma_{1,1} & \cdots & \sigma_{1,r} \\ \sigma_{1,1} & & & \sigma_{1,r+1} \\ \vdots & & & \vdots \\ \sigma_{1,d_1-r} & \cdots & \sigma_{1,d_1-1} & \sigma_{1,d_1} \\ \hline \vdots & & & \vdots \\ \hline \sigma_{m,0} & \sigma_{m,1} & \cdots & \sigma_{m,r} \\ \sigma_{m,1} & & & \sigma_{r+1} \\ \vdots & & & \vdots \\ \sigma_{m,d_m-r} & \cdots & \sigma_{m,d_m-1} & \sigma_{m,d_m} \end{array}\right] \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_r \end{bmatrix} = 0$$

of the form $p = c \prod_{k=1}^{r} (\beta_k x_0 - \alpha_k x_1)$ *with* $[\alpha_k : \beta_k]$ *distinct, then*

$$\psi_l = \sum_{i=1}^{r} \omega_{i,l} (\alpha_l x_0 + \beta_l x_1)^{d_l}$$

for $\omega_{i,l} \in \overline{\mathbb{K}}$ *and* $l = 1, \ldots, m$.

Duality

Dual of polynomial rings

For $R = \mathbb{K}[\mathbf{x}] = \mathbb{K}[x_1, \ldots, x_n] = \{p = \sum_{\alpha \in A} p_\alpha \mathbf{x}^\alpha, p_\alpha \in \mathbb{K}\}$,

$$\mathbb{K}[\mathbf{x}]^* = \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}[\mathbf{x}], \mathbb{K})$$

The element $\sigma \in R^* : p \in R \mapsto \langle \sigma | p \rangle \in \mathbb{K}$ is a **linear functional** on R.

The coefficients $\langle \sigma | \mathbf{x}^\alpha \rangle = \sigma_\alpha \in \mathbb{K}$, $\alpha \in \mathbb{N}^n$ are the **moments** of σ.

Examples:

- $p \mapsto$ coefficient of $\mathbf{x}^\alpha$ in p
- $\mathfrak{e}_\zeta : p \mapsto p(\zeta)$ for $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{K}^n$.
- For $\mathbb{K} = \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ compact, $\int_\Omega : p \mapsto \int_\Omega p(\mathbf{x}) d\mathbf{x}$

Structure of $\mathbb{K}[x]$-module:

$$p \star \sigma \in R^* : q \mapsto \langle \sigma | p\, q \rangle.$$

Example: For $p, q \in R$, $p \star \mathfrak{e}_\zeta : q \mapsto \langle \mathfrak{e}_\zeta | p\, q \rangle = p(\zeta) \langle \mathfrak{e}_\zeta | q \rangle \Rightarrow p \star \mathfrak{e}_\zeta = p(\zeta) \mathfrak{e}_\zeta$

Property: For $p, q \in R$, $\sigma \in R^*$, $p \star (q \star \sigma) = p\, q \star \sigma = q \star (p \star \sigma)$.

Linear functionals as sequences

Correspondence: $\sigma \in \mathbb{K}[\mathbf{x}]^* \equiv (\sigma_\alpha)_{\alpha \in \mathbb{N}^n} \in \mathbb{K}^{\mathbb{N}^n}$ sequence indexed by $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ with $\sigma_\alpha = \langle \sigma | \mathbf{x}^\alpha \rangle$.

$$\sigma : p = \sum_\alpha p_\alpha \mathbf{x}^\alpha \in R \mapsto \langle \sigma | p \rangle = \sum_\alpha \sigma_\alpha p_\alpha \in \mathbb{K}$$

Example: $\mathfrak{e}_\zeta \equiv (\zeta^\alpha)_{\alpha \in \mathbb{K}^{\mathbb{N}^n}}$ where $\zeta^\alpha = \zeta_1^{\alpha_1} \cdots \zeta_n^{\alpha_n}$.

Structure of $\mathbb{K}[x]$-module:

For $p = \sum_{\alpha \in A} p_\alpha \mathbf{x}^\alpha \in R$, $\sigma \equiv (\sigma_\alpha)_{\alpha \in \mathbb{N}^n} \in \mathbb{K}^{\mathbb{N}^n}$, $\beta \in \mathbb{N}^n$

$$(p \star \sigma)_\beta = \sum_{\alpha \in A} p_\alpha \sigma_{\alpha + \beta}$$

(correlation sequence).

Linear functionals as series

Correspondence: $\sigma \in \mathbb{K}[\mathbf{x}]^* \equiv$

$\sigma(\mathbf{y}) = \sum_{\alpha\in\mathbb{N}^n} \sigma_\alpha \frac{\mathbf{y}^\alpha}{\alpha!} \in \mathbb{K}[[y_1,\ldots,y_n]]$ $\quad$ $\sigma(\mathbf{z}) = \sum_{\alpha\in\mathbb{N}^n} \sigma_\alpha \mathbf{z}^\alpha \in \mathbb{K}[[\mathbf{z}_1,\ldots,\mathbf{z}_n]]$

with $\sigma_\alpha = \langle\sigma|\mathbf{x}^\alpha\rangle$, $\alpha! = \prod \alpha_i!$ for $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}^n$.

Example:

$\mathfrak{e}_\zeta(\mathbf{y}) = \sum_\alpha \zeta^\alpha \frac{\mathbf{y}^\alpha}{\alpha!} = e^{\zeta\cdot\mathbf{y}} \in \mathbb{K}[[\mathbf{y}]]$ $\quad$ $\mathfrak{e}_\zeta(\mathbf{z}) = \sum_\alpha \zeta^\alpha \mathbf{z}^\alpha = \frac{1}{\prod_{i=1}^n (1-\zeta_i z_i)} \in \mathbb{K}[[\mathbf{z}]]$

- For $p = \sum_\alpha p_\alpha \in R$, $\sigma(\mathbf{y}) = \sum_{\alpha\in\mathbb{N}^n} \sigma_\alpha \frac{\mathbf{y}^\alpha}{\alpha!} \in \mathbb{K}[[\mathbf{y}]]$, $\langle\sigma|p\rangle = \sum_\alpha \sigma_\alpha p_\alpha$
- The basis dual to $(\mathbf{x}^\alpha)$ is $(\frac{\mathbf{y}^\alpha}{\alpha!})_{\alpha\in\mathbb{N}^n}$ (resp. $(\mathbf{z}^\alpha)_{\alpha\in\mathbb{N}^n}$)
- For $p \in R$, $\alpha \in \mathbb{N}^n$, $\langle\mathbf{y}^\alpha|p\rangle = \partial_{\mathbf{x}}^\alpha(p)(0)$, $\langle\mathbf{z}^\alpha|p\rangle =$ coeff. of $\mathbf{x}^\alpha$ in p.

Structure of R-module:

$$\begin{aligned} x_1 \star \sigma(\mathbf{y}) &= \textstyle\sum_{\alpha_1>0} \sigma_\alpha \frac{\mathbf{y}^{\alpha-e_1}}{(\alpha-e_1)!} & \mathbf{x}_1 \star \sigma(\mathbf{z}) &= \textstyle\sum_{\alpha_1>0} \sigma_\alpha \mathbf{z}^{\alpha-e_1} \\ &= \partial_{y_1}(\sigma(\mathbf{y})) & &= \pi_+(\mathbf{z}_1^{-1}\sigma(\mathbf{z})) \\ p\star\sigma &= p(\partial_1,\ldots,\partial_n)(\sigma)(\mathbf{y}) & \mathbf{p}\star\sigma &= \pi_+(p(z_1^{-1},\ldots,z_n^{-1})\sigma(\mathbf{z})) \end{aligned}$$

Inverse systems

For I an ideal in $R = \mathbb{K}[\mathbf{x}]$,

$$I^\perp = \{\sigma \in R^* \mid \forall p \in I,\ \langle \sigma | p \rangle = 0\}.$$

Dual of quotient algebra: for $\mathcal{A} = R/I$, $\mathcal{A}^* \equiv I^\perp$.

- In $\mathbb{K}[[\mathbf{y}]]$, $I^\perp$ is stable by **derivations** with respect to y_i.
- In $\mathbb{K}[[\mathbf{z}]]$, $I^\perp$ is stable by **"division"** by variables z_i.

Inverse system generated by $\omega_1, \ldots, \omega_r \in \mathbb{K}[\mathbf{y}]$

$$\langle\langle \omega_1, \ldots, \omega_r \rangle\rangle = \langle \partial_{\mathbf{y}}^\alpha(\omega_i), \alpha \in \mathbb{N}^n \rangle \quad \text{resp. } \langle \pi_+(\mathbf{z}^{-\alpha}\omega_{\mathbf{i}}(\mathbf{z})), \alpha \in \mathbb{N}^{\mathbf{n}} \rangle$$

Example: $I = (x_1^2, x_2^2) \subset \mathbb{K}[x_1, x_2]$

$$I^\perp = \langle 1, y_1, y_2, y_1 y_2 \rangle = \langle\langle y_1 y_2 \rangle\rangle \quad \text{resp. } \langle \mathbf{1}, \mathbf{z_1}, \mathbf{z_2}, \mathbf{z_1 z_2} \rangle = \langle\langle \mathbf{z_1 z_2} \rangle\rangle$$

Artinian algebra

Structure of an Artinian algebra $\mathcal{A}$

Definition: $\mathcal{A} = \mathbb{K}[\mathbf{x}]/I$ is **Artinian** if $\dim_{\mathbb{K}} \mathcal{A} < \infty$.

Hilbert nullstellensatz: $\mathcal{A} = \mathbb{K}[\mathbf{x}]/I$ Artinian $\Leftrightarrow \mathcal{V}_{\overline{\mathbb{K}}}(I) = \{\xi_1, \ldots, \xi_r\}$ is finite.

Assuming $\mathbb{K} = \overline{\mathbb{K}}$ is algebraically closed, we have

- $I = Q_1 \cap \cdots \cap Q_r$ where Q_i is m_{ξ_i}-primary where $\mathcal{V}_{\overline{\mathbb{K}}}(I) = \{\xi_1, \ldots, \xi_r\}$.
- $\mathcal{A} = \mathbb{K}[\mathbf{x}]/I = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_r$, with
  - $\mathcal{A}_i = \mathbf{u}_i\, \mathcal{A} \sim \mathbb{K}[x_1, \ldots, x_n]/Q_i$,
  - $\mathbf{u}_i^2 = \mathbf{u}_i$, $\mathbf{u}_i\, \mathbf{u}_j = 0$ if $i \neq j$, $\mathbf{u}_1 + \cdots + \mathbf{u}_r = 1$.
- $\dim R/Q_i = \mu_i$ is the multiplicity of ξ_i.

Structure of the dual $\mathcal{A}^*$

Sparse series:

$$\mathcal{P}ol\mathcal{E}xp = \left\{ \sigma(\mathbf{y}) = \sum_{i=1}^{r} \omega_i(\mathbf{y})\, \mathfrak{e}_{\xi_i}(\mathbf{y}) \mid \omega_i(\mathbf{y}) \in \mathbb{K}[\mathbf{y}], \right\}$$

where $\mathfrak{e}_{\xi_i}(\mathbf{y}) = e^{\mathbf{y}\cdot\xi_i} = e^{y_1\xi_{1,i}+\cdots+y_n\xi_{n,i}}$ with $\xi_{i,j} \in \mathbb{K}$.

Inverse system generated by $\omega_1, \ldots, \omega_r \in \mathbb{K}[\mathbf{y}]$

$$\langle\langle \omega_1, \ldots, \omega_r \rangle\rangle = \langle \partial_{\mathbf{y}}^{\alpha}(\omega_i), \alpha \in \mathbb{N}^n \rangle$$

Theorem

For $\mathbb{K} = \overline{\mathbb{K}}$ algebraically closed,

$$\mathcal{A}^* = \oplus_{i=1}^{r} \mathcal{D}_i\, \mathfrak{e}_{\xi_i}(\mathbf{y}) \quad \subset \quad \mathcal{P}ol\mathcal{E}xp$$

- $\mathcal{V}_{\overline{\mathbb{K}}}(I) = \{\xi_1, \ldots, \xi_r\}$
- $\mathcal{D}_i = \langle\langle \omega_{i,1}, \ldots, \omega_{i,l_i} \rangle\rangle$ *with* $\omega_{i,j} \in \mathbb{K}[\mathbf{y}]$, $Q_i^{\perp} = \mathcal{D}_i \mathfrak{e}_{\xi}$ *where* $I = Q_1 \cap \cdots \cap Q_r$
- $\mu(\omega_{i,1}, \ldots, \omega_{i,l_i}) := \dim_{\mathbb{K}}(\mathcal{D}_i) = \mu_i$ *multiplicity of* ξ_i.

The roots by eigencomputation

Hypothesis: $\mathcal{V}_{\mathbb{K}}(I) = \{\xi_1, \ldots, \xi_r\} \Leftrightarrow \mathcal{A} = \mathbb{K}[\mathbf{x}]/I$ Artinian.

$$\begin{array}{rcl} \mathcal{M}_a : \mathcal{A} & \to & \mathcal{A} \\ u & \mapsto & a\,u \end{array} \qquad \begin{array}{rcl} \mathcal{M}_a^{\mathsf{t}} : \mathcal{A}^* & \to & \mathcal{A}^* \\ \Lambda & \mapsto & a \star \Lambda = \Lambda \circ \mathcal{M}_a \end{array}$$

Theorem

- *The eigenvalues of $\mathcal{M}_a$ are $\{a(\xi_1), \ldots, a(\xi_r)\}$.*
- *The eigenvectors of all $(\mathcal{M}_a^{\mathsf{t}})_{a \in \mathcal{A}}$ are (up to a scalar) $\mathfrak{e}_{\xi_i} : p \mapsto p(\xi_i)$.*

Proposition

If the roots are simple, the operators $\mathcal{M}_a$ are diagonalizable. Their common eigenvectors are, up to a scalar, **interpolation polynomials** $\mathbf{u}_i$ *at the roots and idempotent in $\mathcal{A}$.*

Theorem

In a basis of $\mathcal{A}$, all the matrices M_a ($a \in \mathcal{A}$) are of the form

$$\mathtt{M}_a = \begin{bmatrix} \mathtt{N}_a^1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathtt{N}_a^r \end{bmatrix} \text{ with } \mathtt{N}_a^i = \begin{bmatrix} a(\xi_i) & & \star \\ & \ddots & \\ \mathbf{0} & & a(\xi_i) \end{bmatrix}$$

Corollary (Chow form)

$\Delta(\mathbf{u}) = \det(v_0 + v_1\,\mathtt{M}_{x_1} + \cdots + v_n\,\mathtt{M}_{x_n}) = \prod_{i=1}^{r}(v_0 + v_1\xi_{i,1} + \cdots + v_n\xi_{i,n})^{\mu_{\xi_i}}$ *where* μ_{ξ_i} *is the multiplicity of* ξ.

Example

Roots of polynomial systems

$$\begin{cases} f_1 &= x_1^2 x_2 - x_1^2 \\ f_2 &= x_1 x_2 - x_2 \end{cases} \qquad I = (f_1, f_2) \subset \mathbb{C}[\mathbf{x}]$$

$$\mathcal{A} = \mathbb{C}[\mathbf{x}]/I \equiv \langle 1, x_1, x_2 \rangle \quad I = (x_1^2 - x_2, x_1 x_2 - x_2, x_2^2 - x_2)$$

$$M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad \begin{matrix} \text{common} \\ \text{eigvecs of} \\ M_1^t, M_2^t \end{matrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$I = Q_1 \cap Q_2 \quad \text{where} \quad Q_1 = (x_1^2, x_2), \quad Q_2 = \mathbf{m}_{(1,1)} = (x_1 - 1, x_2 - 1)$$

$$I = Q_1^\perp \oplus Q_2^\perp \quad Q_1^\perp = \langle 1, y_1 \rangle = \langle 1, y_1 \rangle\, \mathfrak{e}_{(0,0)}(\mathbf{y}) \quad Q_2^\perp = \langle 1 \rangle\, \mathfrak{e}_{(1,1)}(\mathbf{y}) = \langle e^{y_1 + y_2} \rangle$$

Solution of partial differential equations (with constant coeff.)

$$\begin{cases} \partial_{y_1}^2 \partial_{y_2} \sigma - \partial_{y_1}^2 \sigma &= 0 \qquad f_1 \star \sigma = 0 \\ \partial_{y_1} \partial_{y_2} \sigma - \partial_{y_2} \sigma &= 0 \qquad f_2 \star \sigma = 0 \end{cases} \quad \Rightarrow \quad \sigma \in I^\perp = Q_1^\perp \oplus Q_2^\perp$$

$$\sigma = a + b\, y_1 + c\, e^{y_1 + y_2} \qquad a, b, c \in \mathbb{C}$$

Solving by duality

To find the roots $\mathcal{V}(I)$, we compute the structure of $\mathcal{A} = R/I$, that is,

- a vector space $B \subset R$ spanned by a "basis" of $\mathcal{A}$,
- the multiplication operators M_i by variables x_i in the basis of B.

We use a **normal form** $\mathcal{N}$ on R w.r.t. I, that is a projector $\mathcal{N} : R \to B$ s.t. $\ker \mathcal{N} = I$ and $\mathcal{N}_{|B} = \mathrm{Id}_B$.

The operators M_i are given by $M_i : b \in B \mapsto \mathcal{N}(x_i b) \in B$.

Classical examples:

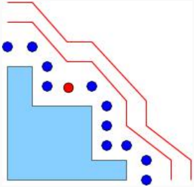
- $\mathcal{N} : p \in \mathbb{K}[x] \mapsto$ remainder of p in the Euclidean division by f where $I = (f) \subset \mathbb{K}[x]$.
- $\mathcal{N} : p \in R \mapsto$ remainder of p in the reduction by a Grobner basis.

Truncated Normal Forms (TNF)

☞ **If B is known, we only need to know $\mathcal{N}$ on $B^+ = B + x_1 B + \cdots + x_n B$, to know the operators of multiplication M_i.**

For $B \subset V \subset R$ with $x_i \cdot B \subset V, i = 1, \ldots, n$, a **Truncated Normal Form** on V w.r.t. I is a projector $\mathcal{N} : V \to B$ such that $\ker \mathcal{N} = I \cap V$ and $\mathcal{N}_{|B} = \mathrm{Id}_B$.

Border basis



If B is spanned by a set of monomials $\mathcal{B}$, $V = \langle \mathcal{B}^+ \rangle$, and $\partial\mathcal{B} = \mathcal{B}^+ \setminus \mathcal{B}$, we consider projections of $\mathbf{x}^\alpha \in \partial\mathcal{B}$

Definition (Border basis)

$$f_\alpha = \mathbf{x}^\alpha - \sum_{\mathbf{x}^\beta \in \mathcal{B}} c_{\alpha,\beta}\mathbf{x}^\beta \quad \alpha \in \partial\mathcal{B}$$

such that $N : \mathbf{x}^\beta \in \mathcal{B}^+ \mapsto \begin{cases} \mathbf{x}^\beta & \text{if } \mathbf{x}^\beta \in \mathcal{B} \\ \mathbf{x}^\beta - f_\beta & \text{if } \mathbf{x}^\beta \in \partial\mathcal{B} \end{cases}$ is a **TNF**.

If $F = (f_\alpha)_{\alpha \in \partial\mathcal{B}}$ is a border basis,

$$R = B \oplus (F)$$

and the projection on B along (F) is a normal form $\mathcal{N}$, which extends N.

Definition: V connected to 1 if $V_0 = \langle 1 \rangle \subset V_1 \subset \cdots \subset V_s = V$ with $V_{i+1} \subset V_i^+$.

For $F \subset R$, let $\mathrm{Com}_V(F)$ (**commutation polynomials**) be the set of polynomials in V of the form $x_i f$ or $x_i f - x_j f'$ with $f, f' \in F$, $i \neq j$.

Theorem

Let $B, V \subset R$ such that $W := B^+ \subset V$, V is connected to 1 and let $N : V \to B$ be a projector such that $F := \ker N \subset I \cap V$ and $M_i : b \in B \mapsto N(x_i b) \in B$. Then the following points are equivalent:

1. $(M_i \circ M_j - M_j \circ M_i) = 0$ *for* $1 \leq i, j \leq n$;
2. *there exists a unique normal form* $\mathcal{N} : R \to B$ *s.t.* $\mathcal{N}_{|V} = N$ *and* $\ker \mathcal{N} = (F)$;
3. $F^+ \cap W \subset F$;
4. $\mathrm{Com}_W(F) \subset F$;

☞ **Algorithm to compute a border basis by adding to F the non-zero reduction of the commutation polynomials of F [MT05,MT08,...].**

Dual description

A TNF $N : V \to B$ modulo I with B of dimension r is given by $N : f \in V \to N(f) = (\eta_1(f), \ldots, \eta_r(f)) \in \mathbb{K}^r$ where

$$\eta_i \in V^* \cap I^\perp = \{\sigma \in V^* \mid \forall p \in I \cap V, \sigma(p) = 0\}.$$

Theorem (TMV18)

Let $V \subset R$ be a finite dimensional, $W \subset V$ s.t. $W^+ \subset V$ and $N : V \to \mathbb{K}^r$ s.t.

1. *$\exists u \in V$ such that $u + I$ is a unit in R/I,*
2. *$\ker(N) \subset I \cap V$,*
3. *$N_{|W}$ is onto $\mathbb{K}^r$.*

Then for any r-dimensional vector subspace $B \subset W$ s.t. $N_{|B}$ is invertible we have:

(i) *$B \simeq R/I$ (as R-modules),*

(ii) *$V = B \oplus (I \cap V)$ and $I = (\langle \ker(N) \rangle : u)$, N is a TNF,*

(iii) *$M_i : b \in B \mapsto N(x_i b) \in B$ is the multiplication by x_i in B modulo I.*

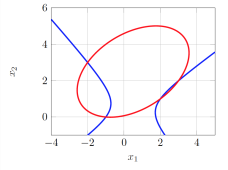
Algorithm

For $f_1, \dots, f_m \in R$, $V_1, \dots, V_m, V$ vector spaces of R (e.g. spanned by monomials)

$$\begin{array}{rccc} \mathrm{Res}: & V_1 \times \cdots \times V_m & \longrightarrow & V \\ & (q_1, \dots, q_n) & \longmapsto & q_1 f_1 + \cdots + q_m f_m. \end{array}$$

Roots from the cokernel of a resultant map

- $N \leftarrow (\ker \mathrm{Res}^t)^t$
- $N_{|W} \leftarrow$ restriction of N to W with $W^+ \subset V$
- $\mathtt{Q}, \mathtt{R}, \mathtt{P} \leftarrow \mathit{qrfact}(N_{|W})$
  $N_0 \leftarrow$ first columns in $\mathtt{P}$ of $N_{|W}$ indexed by $B \subset W$
- $N_i \leftarrow$ columns of N corresponding to $x_i \cdot B$
- $M_{x_i} \leftarrow (N_0)^{-1} N_i$
- **return** the roots of $f_1, \dots, f_m$ from $M_{x_1}, \dots, M_{x_n}$.



Consider the ideal $I = \langle f_1, f_2 \rangle \subset \mathbb{C}[x_1, x_2]$ given by

$$\begin{aligned} f_1 &= 7 + 3x_1 - 6x_2 - 4x_1^2 + 2x_1x_2 + 5x_2^2, \\ f_2 &= -1 - 3x_1 + 14x_2 - 2x_1^2 + 2x_1x_2 - 3x_2^2. \end{aligned}$$

$$\mathrm{Res}^\top = \begin{array}{c} \\ f_1 \\ x_1 f_1 \\ x_2 f_1 \\ f_2 \\ x_1 f_2 \\ x_2 f_2 \end{array} \begin{array}{c} \begin{array}{cccccccccc} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 & x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 \end{array} \\ \left[\begin{array}{cccccccccc} 7 & 3 & -6 & -4 & 2 & 5 & & & & \\ & 7 & & 3 & -6 & & -4 & 2 & 5 & \\ & & 7 & & 3 & -6 & & -4 & 2 & 5 \\ -1 & -3 & 14 & -2 & 2 & -3 & & & & \\ & -1 & & -3 & 14 & & -2 & 2 & -3 & \\ & & -1 & & -3 & 14 & & -2 & 2 & -3 \end{array}\right] \end{array}.$$

We compute $\ker \mathrm{Res}^\top$ and find linear functionals $\eta_i, i = 1, \ldots, 4$ in $V^* \cap I^\perp$ (representing $\mathfrak{e}_{\xi_i}$):

$$N = \begin{array}{c} \\ v^{(3)}(-2,3) \\ v^{(3)}(3,2) \\ v^{(3)}(2,1) \\ v^{(3)}(-1,0) \end{array} \begin{array}{c} \begin{array}{cccccccccc} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 & x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 \end{array} \\ \begin{bmatrix} 1 & -2 & 3 & 4 & -6 & 9 & -8 & 12 & -18 & 27 \\ 1 & 3 & 2 & 9 & 6 & 4 & 27 & 18 & 12 & 8 \\ 1 & 2 & 1 & 4 & 2 & 1 & 8 & 4 & 2 & 1 \\ 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

For $B = \{x_1, x_2, x_1^2, x_1x_2\}$, the submatrices we need are

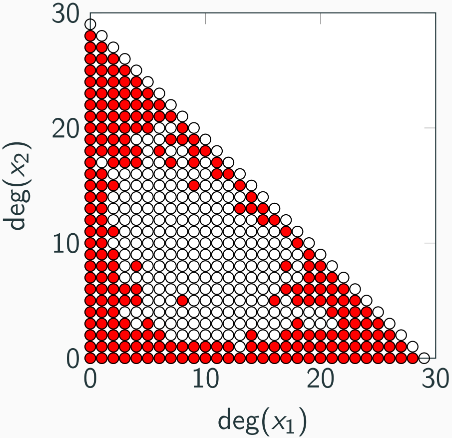
$$N_{|B} = \begin{bmatrix} -2 & 3 & 4 & -6 \\ 3 & 2 & 9 & 6 \\ 2 & 1 & 4 & 2 \\ -1 & 0 & 1 & 0 \end{bmatrix},\ N_1 = \begin{bmatrix} 4 & -6 & -8 & 12 \\ 9 & 6 & 27 & 18 \\ 4 & 2 & 8 & 4 \\ 1 & 0 & -1 & 0 \end{bmatrix},\ N_2 = \begin{bmatrix} -6 & 9 & 12 & -18 \\ 6 & 4 & 18 & 12 \\ 2 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We obtain the solutions $\xi_1 = (-2, 3), \xi_2 = (3, 2), \xi_3 = (2, 1), \xi_4 = (-1, 0)$ by eigen computation.

Example of basis for a generic dense system

A system $f_1, f_2 \in R = \mathbb{R}[x_1, x_2]$ with $\deg(f_i) = 15$

$V = R_{\leq 29}$, $W = R_{\leq 28}$, $\delta = 225$



Numerical experimentation

$n = 2$, numerical quality and running time.

d	δ	m_1	$m_2 = n_1$	n_2	res	δ_{alg}	δ_{phc}	δ_{brt}
1	1	2	3	1	$1.28 \cdot 10^{-16}$	1	1	1
7	49	56	105	49	$2.06 \cdot 10^{-13}$	49	49	49
13	169	182	351	169	$2.18 \cdot 10^{-13}$	169	169	169
19	361	380	741	361	$5.28 \cdot 10^{-13}$	361	361	361
25	625	650	1,275	625	$1.21 \cdot 10^{-10}$	625	614	625
31	961	992	1,953	961	$5.23 \cdot 10^{-9}$	961	951	961
37	1,369	1,406	2,775	1,369	$4.05 \cdot 10^{-12}$	1,369	1,360	1,368
43	1,849	1,892	3,741	1,849	$1.74 \cdot 10^{-11}$	1,849	1,825	1,845
49	2,401	2,450	4,851	2,401	$1.57 \cdot 10^{-10}$	2,401	2,364	2,163
55	3,025	3,080	6,105	3,025	$1.84 \cdot 10^{-11}$	3,025	2,970	2,487
61	3,721	3,782	7,503	3,721	$3.26 \cdot 10^{-11}$	3,721	3,662	2,260

d	t_M	t_N	t_B	t_S	t_{alg}	t_{phc}	t_{brt}
1	$1.48 \cdot 10^{-4}$	$5.5 \cdot 10^{-5}$	$2.96 \cdot 10^{-4}$	$3.6 \cdot 10^{-5}$	$5.35 \cdot 10^{-4}$	$5.6 \cdot 10^{-2}$	$1.41 \cdot 10^{-2}$
7	$7.88 \cdot 10^{-3}$	$1.68 \cdot 10^{-3}$	$3.76 \cdot 10^{-3}$	$2.78 \cdot 10^{-3}$	$1.61 \cdot 10^{-2}$	0.18	$8.65 \cdot 10^{-2}$
13	$4.65 \cdot 10^{-2}$	$1.03 \cdot 10^{-2}$	$1.66 \cdot 10^{-2}$	$2.81 \cdot 10^{-2}$	0.1	0.84	1.14
19	0.13	$5.69 \cdot 10^{-2}$	$5.34 \cdot 10^{-2}$	0.13	0.37	3.29	8.79
25	0.32	0.18	0.15	0.51	1.16	8.79	33.83
31	0.55	0.51	0.55	1.49	3.1	20.25	98.39
37	0.96	1.52	1.5	3.52	7.5	39.92	258.09
43	1.47	4.05	3.8	8.28	17.6	69.1	504.01
49	2.47	10.46	8.78	17.91	39.62	124.47	891.37
55	3.69	20.51	17.85	34.3	76.34	178.55	1,581.77
61	4.85	36.32	31.26	62.87	135.3	283.87	2,115.66

$n = 3$, numerical quality and running time.

d	δ	m_1	$m_2 = n_1$	n_2	res	δ_{alg}	δ_{phc}	δ_{brt}
1	1	3	4	1	$1.79 \cdot 10^{-16}$	1	1	1
3	27	105	120	27	$1.05 \cdot 10^{-14}$	27	27	27
5	125	495	560	125	$1.29 \cdot 10^{-12}$	125	125	125
7	343	1,365	1,540	343	$6.71 \cdot 10^{-12}$	343	343	343
9	729	2,907	3,276	729	$1.38 \cdot 10^{-10}$	729	726	729
11	1,331	5,313	5,984	1,331	$3.11 \cdot 10^{-11}$	1,331	1,331	1,331
13	2,197	8,775	9,880	2,197	$2.86 \cdot 10^{-11}$	2,197	2,192	2,197

d	t_M	t_N	t_B	t_S	t_{alg}	t_{phc}	t_{brt}
1	$3.72 \cdot 10^{-4}$	$1.24 \cdot 10^{-4}$	$2.31 \cdot 10^{-3}$	$4.5 \cdot 10^{-5}$	$2.85 \cdot 10^{-3}$	$6.8 \cdot 10^{-2}$	$1.69 \cdot 10^{-2}$
3	$7.91 \cdot 10^{-3}$	$2.42 \cdot 10^{-3}$	$7.06 \cdot 10^{-3}$	$1.08 \cdot 10^{-3}$	$1.85 \cdot 10^{-2}$	0.14	$7.33 \cdot 10^{-2}$
5	$5.66 \cdot 10^{-2}$	$3.93 \cdot 10^{-2}$	$3.31 \cdot 10^{-2}$	$1.17 \cdot 10^{-2}$	0.14	0.68	0.63
7	0.23	1.13	0.12	$9.9 \cdot 10^{-2}$	1.57	3.42	4.11
9	0.68	14.43	0.65	0.63	16.4	12.21	17.29
11	1.77	44.79	3.91	3.98	54.46	39.08	70.66
13	5.81	183.67	16.07	15.35	220.9	97.28	210.34

Decomposition algorithms

Hankel operators

Hankel operator: For $\sigma = (\sigma_1, \ldots, \sigma_m) \in (R^*)^m$,

$$\begin{array}{rcl} H_\sigma : R & \to & (R^*)^m \\ p & \mapsto & (p \star \sigma_1, \ldots, p \star \sigma_m) \end{array}$$

σ is the **symbol** of H_σ.

Truncated Hankel operator: $V, W_1, \ldots, W_m \subset R$,

$$H_\sigma^{V,W} : p \in V \to ((p \star \sigma_i)_{|W_i})$$

Property: $V = \langle \mathbf{x}^\alpha \rangle_{\alpha \in A} = \langle \mathbf{x}^A \rangle$, $W = \langle \mathbf{x}^\beta \rangle_{\beta \in B} = \langle \mathbf{x}^B \rangle \subset R$, $\sigma \in R^*$,

$$H_\sigma^{A,B} = [\langle \sigma | \mathbf{x}^\alpha \mathbf{x}^\beta \rangle]_{\alpha \in A, \beta \in B} = [\sigma_{\alpha+\beta}]_{\alpha \in A, \beta \in B}.$$

Example: $m = 1$, $\sigma = (0, 1, 1, 2, 3, 5, 8, 13, \ldots)$.

For $B = \{1, x, x^2\}$,

$$H_\sigma^{B,B} = (\sigma_{i+j})_{0 \le i,j \le 2} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

Ideal: $I_\sigma = \ker H_\sigma$

$$I_\sigma = \{p \in \mathbb{K}[\mathbf{x}] \mid p \star \sigma = 0\},$$
$$= \{p = \sum_\alpha p_\alpha \mathbf{x}^\alpha \mid \forall \beta \in \mathbb{N}^n \sum_\alpha p_\alpha \sigma_{\alpha+\beta} = 0\} \text{ (Linear recurrence relations)}$$

Quotient algebra: $\mathcal{A}_\sigma = R/I_\sigma$

☞ $\sigma \in \mathcal{A}_\sigma^* = I_\sigma^\perp$ ($p \star \sigma = 0$ implies $\langle \sigma | p \rangle = 0$).

Compute the decomposition of σ by analyzing the structure of $\mathcal{A}_\sigma^*$.

Example of Fibonnaci sequence: $\sigma = (0, 1, 1, 2, 3, 5, 8, 13, \ldots)$

$$H_\sigma = \begin{pmatrix} 0 & 1 & 1 & 2 & \ldots \\ 1 & 1 & 2 & 3 & \ldots \\ 1 & 2 & 3 & 5 & \ldots \\ 2 & 3 & 5 & 8 & \ldots \\ \vdots & \vdots & \vdots & & \end{pmatrix} \qquad H_\sigma \begin{pmatrix} \vdots \\ -1 \\ -1 \\ 1 \\ \vdots \end{pmatrix} = 0$$

$I_\sigma = \ker H_\sigma = (x^2 - x - 1)$.

$\mathcal{A}_\sigma = \mathbb{K}[x]/(x^2 - x - 1)$ with basis $\{1, x\}$.

Multiplication by x in this basis of $\mathcal{A}_\sigma$: $M_x = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$.

Eigenvalues: $\xi_i = \frac{1+(-1)^{i+1}\sqrt{5}}{2}$. Eigenvectors: $\mathbf{u}_i = \frac{(-1)^{i+1}}{\sqrt{5}}(x - \xi_i)$, $i = 1, 2$.

Matrix of $\overline{H}_\sigma$ in this basis: $\overline{H}_\sigma = \begin{pmatrix} \frac{1}{\sqrt{5}} & 0 \\ 0 & -\frac{1}{\sqrt{5}} \end{pmatrix}$.

Univariate series:

Kronecker (1881)

The Hankel operator

$$\begin{array}{rcl} H_\sigma : \mathbb{C}^{\mathbb{N},finite} & \to & \mathbb{C}^{\mathbb{N}} \\ (p_m) & \mapsto & (\sum_m \sigma_{m+n} p_m)_{n\in\mathbb{N}} \end{array}$$

is of **finite rank** r iff $\exists \omega_1, \ldots, \omega_{r'} \in \mathbb{C}[y]$ and $\xi_1, \ldots, \xi_{r'} \in \mathbb{C}$ distincts s.t.

$$\sigma(y) = \sum_{n\in\mathbb{N}} \sigma_n \frac{y^n}{n!} = \sum_{i=1}^{r'} \omega_i(y) \mathbf{e}_{\xi_i}(y)$$

with $\sum_{i=1}^{r'} (\deg(\omega_i) + 1) = r$.

Multivariate series:

Theorem (Generalized Kronecker Theorem)

For $\sigma = (\sigma_1, \ldots, \sigma_m) \in (R^*)^m$, the Hankel operator

$$\begin{array}{rcl} H_\sigma : R & \rightarrow & (R^*)^m \\ p & \mapsto & (p \star \sigma_1, \ldots, p \star \sigma_m) \end{array}$$

is of rank r iff

$$\sigma_j = \sum_{i=1}^{r'} \omega_{j,i}(\mathbf{y})\, \mathfrak{e}_{\xi_i}(\mathbf{y}) \in \mathcal{P}ol\mathcal{E}xp, \quad j = 1, \ldots, m$$

with $r = \sum_{i=1}^{r'} \mu(\omega_{1,i}, \ldots, \omega_{m,i})$. In this case, we have

- $\mathcal{V}_{\mathbb{C}}(I_\sigma) = \{\xi_1, \ldots, \xi_{r'}\}$.
- $I_\sigma = Q_1 \cap \cdots \cap Q_{r'}$ *with* $Q_i^\perp = \langle\langle \omega_{1,i}, \ldots, \omega_{m,i} \rangle\rangle\, \mathfrak{e}_{\xi_i}(\mathbf{y})$.

If $m = 1$, $\mathcal{A}_\sigma$ is **Gorenstein** *($\mathcal{A}_\sigma^* = \mathcal{A}_\sigma \star \sigma$ is a free $\mathcal{A}_\sigma$-module of rank 1) and $(a, b) \mapsto \langle \sigma | ab \rangle$ is non-degenerate in $\mathcal{A}_\sigma$.*

Decomposition from the structure of $\mathcal{A}_\sigma$

For $\sigma \in (R^*)^m$ with $\dim \mathcal{A}_\sigma = r$:

- For B, C be of size r, if $H_\sigma^{B,C}$ is invertible then B is a basis of $\mathcal{A}_\sigma$.
- The matrix M_i of multiplication by x_i in the basis B of $\mathcal{A}_\sigma$ is such that

$$\mathbf{H}_\sigma^{\mathbf{x_i}\mathbf{B},\mathbf{C}} = \mathbf{H}_{\mathbf{x_i}\star\sigma}^{\mathbf{B},\mathbf{C}} = \mathbf{H}_\sigma^{\mathbf{B},\mathbf{C}}\,\mathbf{M_i}$$

- The common **eigenvectors** of M_i^t are (up to a scalar) the vectors $[B(\xi_{\mathbf{i}})]$, $i = 1, \ldots, r$.

For $\sigma = \sum_{i=1}^{\mathbf{r}} \omega_{\mathbf{i}}\, \mathfrak{e}_{\xi_{\mathbf{i}}}$, with $\omega_{\mathbf{i}} \in \mathbb{C} \setminus \{0\}$ and $\xi_i \in \mathbb{C}^n$ distinct.

- rank $H_\sigma = \mathbf{r}$ and the multiplicity of the points $\xi_{\mathbf{1}}, \ldots, \xi_{\mathbf{r}}$ in $\mathcal{V}(I_\sigma)$ is 1.

- The common **eigenvectors** of M_i are (up to a scalar) the Lagrange **interpolation polynomials $\mathbf{u}_{\xi_{\mathbf{i}}}$** at the points $\xi_{\mathbf{i}}$, $i = 1, \ldots, r$.

$$\mathbf{u}_{\xi_{\mathbf{i}}}(\xi_{\mathbf{j}}) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases} \qquad \mathbf{u}_{\xi_{\mathbf{i}}}{}^2 \equiv \mathbf{u}_{\xi_{\mathbf{i}}},\ \sum_{i=1}^{\mathbf{r}} \mathbf{u}_{\xi_{\mathbf{i}}} \equiv 1.$$

Decomposition algorithm

Input: The first coefficients $(\sigma_\alpha)_{\alpha \in A}$ of the series

$$\sigma = \sum_{\alpha \in \mathbb{N}^n} \sigma_\alpha \frac{\mathbf{y}^\alpha}{\alpha!} = \sum_{i=1}^{r} \omega_i \mathfrak{e}_{\xi_i}(\mathbf{y})$$

1. Compute bases $B, B' \subset \langle \mathbf{x}^A \rangle$ s.t. that $H^{B',B}$ invertible and $|B| = |B'| = r = \dim \mathcal{A}_\sigma$;
2. Deduce the tables of multiplications $M_i := (H_\sigma^{B',B})^{-1} H_\sigma^{B',x_i B}$
3. Compute the eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ of $\sum_i l_i M_i$ for a generic $\mathbf{l} = l_1 x_1 + \cdots + l_n x_n$;
4. Deduce the points $\xi_i = (\xi_{i,1}, \ldots, \xi_{i,n})$ s.t. $M_j \mathbf{v}_i - \xi_{i,j} \mathbf{v}_i = 0$ and the weights $\omega_i = \frac{1}{\mathbf{v}_i(\xi_i)} \langle \sigma | \mathbf{v}_i \rangle$.

Output: The decomposition $\sigma = \sum_{i=1}^{r} \frac{1}{\mathbf{v}_i(\xi_i)} \langle \sigma | \mathbf{v}_i \rangle \mathbf{e}_{\xi_i}(\mathbf{y})$.

Multivariate Prony method

Let $h(t_1,t_2) = 2 + 3\,2^{t_1}\,2^{t_2} - 3^{t_1}$, $\sigma = \sum_{\alpha\in\mathbb{N}^2} h(\alpha)\frac{\mathbf{y}^\alpha}{\alpha!} = 2\mathfrak{e}_{(1,1)}(\mathbf{y}) + 3\mathfrak{e}_{(2,2)}(\mathbf{y}) - \mathfrak{e}_{(3,1)}(\mathbf{y})$.

- Take $B = \{1, x_1, x_2\}$ and compute

$$H_0 := H_\sigma^{B,B} = \begin{bmatrix} h(0,0) & h(1,0) & h(0,1) \\ h(1,0) & h(2,0) & h(1,1) \\ h(0,1) & h(1,1) & h(0,2) \end{bmatrix} = \begin{bmatrix} 4 & 5 & 7 \\ 5 & 5 & 11 \\ 7 & 11 & 13 \end{bmatrix},$$

$$H_1 := H_\sigma^{B,x_1 B} = \begin{bmatrix} 5 & 5 & 7 \\ 5 & -1 & 17 \\ 811 & 178 & 23 \end{bmatrix},\ H_2 := H_\sigma^{B,x_2 B} = \begin{bmatrix} 7 & 11 & 13 \\ 11 & 17 & 23 \\ 13 & 23 & 25 \end{bmatrix}.$$

- Compute the generalized eigenvectors of $(aH_1 + bH_2, H_0)$:

$$U = \begin{bmatrix} 2 & -1 & 0 \\ -1/2 & 0 & 1/2 \\ -1/2 & 1 & -1/2 \end{bmatrix} \text{ and } H_0\,U = \begin{bmatrix} 2 & 3 & -1 \\ 2\times 1 & 3\times 2 & -1\times 3 \\ 2\times 1 & 3\times 2 & -1\times 1 \end{bmatrix}.$$

- This yields the weights $2, 3, -1$ and the roots $(1,1), (2,2), (3,1)$.

Demo

A general framework

- $\mathfrak{F}$ the functional space, in which the "signal" lives.
- $S_1, \ldots, S_n : \mathfrak{F} \to \mathfrak{F}$ commuting linear operators: $S_i \circ S_j = S_j \circ S_i$.
- $\Delta : h \in \mathfrak{F} \mapsto \Delta[h] \in \mathbb{C}$ a linear functional on $\mathfrak{F}$.

Generating series associated to $h \in \mathfrak{F}$:

$$\sigma_h(\mathbf{y}) \quad = \quad \sum_{\alpha \in \mathbb{N}^n} \Delta[S^\alpha(h)] \, \frac{\mathbf{y}^\alpha}{\alpha!} = \sum_{\alpha \in \mathbb{N}^n} \sigma_\alpha \frac{\mathbf{y}^\alpha}{\alpha!}.$$

- Eigenfunctions:

$$S_j(E) = \xi_j E, j = 1, \ldots, n \Rightarrow \sigma_E = \omega \, \mathfrak{e}_\xi(\mathbf{y}).$$

- Generalized eigenfunctions:

$$S_j(E_k) = \xi_j E_k + \sum_{k' < k} m_{j,k'} E_{k'} \Rightarrow \sigma_{E_k} = \omega_i(\mathbf{y}) \mathfrak{e}_\xi(\mathbf{y}).$$

☞ **If $h \mapsto \sigma_h$ is injective $\Rightarrow$ unique decomposition of f as a linear**

Sparse reconstruction from Fourier coefficients

- $\mathcal{F} = L^2(\Omega)$;
- $S_i : h(\mathbf{x}) \in L^2(\Omega) \mapsto e^{2\pi \frac{x_i}{T_i}} h(\mathbf{x}) \in L^2(\Omega)$ is the multiplication by $e^{2\pi \frac{x_i}{T_i}}$;
- $\Delta : h(\mathbf{x}) \in \mathcal{O}'_C \mapsto \int h(\mathbf{x}) d\mathbf{x} \in \mathbb{C}$.

The moments of f are

$$\sigma_\gamma = \frac{1}{\prod_{j=1}^n T_j} \int f(\mathbf{x}) e^{-2\pi \mathbf{i} \sum_{j=1}^n \frac{\gamma_j x_j}{T_j}} d\mathbf{x}$$

Eigenfunctions: δ_ξ; generalized eigenfunctions: $\delta_\xi^{(\alpha)}$.

For $f \in L^2(\Omega)$ and $\sigma = (\sigma_\gamma)_{\gamma \in \mathbb{Z}^n}$ its Fourier coefficients,

$$\Gamma_\sigma : (\rho_\beta)_{\beta \in \mathbb{Z}^n} \in L^2(\mathbb{Z}^n) \mapsto \left(\sum_\beta \sigma_{\alpha+\beta} \rho_\beta \right)_{\alpha \in \mathbb{Z}^n} \in L^2(\mathbb{Z}^n).$$

Γ_σ is of finite rank r if and only if $f = \sum_{i=1}^{r'} \sum_{\alpha \in A_i \subset \mathbb{N}^n} \omega_{i,\alpha} \delta_{\xi_i}^{(\alpha)}$ with

Symmetric tensor decomposition and Waring problem (1770)

Symmetric tensor decomposition problem:

Given a homogeneous polynomial ψ of degree d in the variables $\bar{\mathbf{x}} = (x_0, x_1, \ldots, x_n)$ with coefficients $\in \mathbb{K}$:

$$\psi(\bar{\mathbf{x}}) = \sum_{|\alpha|=d} \sigma_\alpha \binom{d}{\alpha} \bar{\mathbf{x}}^\alpha,$$

find a minimal decomposition of ψ of the form

$$\psi(\bar{\mathbf{x}}) = \sum_{i=1}^{r} \omega_i (\xi_{i,0} x_0 + \xi_{i,1} x_1 + \cdots + \xi_{i,n} x_n)^d$$

with $\xi_i = (\xi_{i,0}, \xi_{i,1}, \ldots, \xi_{i,n}) \in \overline{\mathbb{K}}^{n+1}$ spanning disctint lines, $\omega_i \in \overline{\mathbb{K}}$.

The minimal r in such a decomposition is called the **rank** of ψ.

Symmetric tensors and apolarity

Apolar product: For $f = \sum_{|\alpha|=d} f_\alpha \binom{d}{\alpha} \overline{\mathbf{x}}^\alpha$, $g = \sum_{|\alpha|=d} g_\alpha \binom{d}{\alpha} \overline{\mathbf{x}}^\alpha \in \mathbb{K}[\overline{\mathbf{x}}]_d$,

$$\langle f, g \rangle_d = \sum_{|\alpha|=d} f_\alpha \, g_\alpha \binom{d}{\alpha}.$$

Property: $\langle f, (\xi_0 x_0 + \cdots + \xi_n x_n)^d \rangle = f(\xi_0, \ldots, \xi_n)$

Duality: For $\psi \in S_d$, we define $\psi^* \in S_d^* = \mathrm{Hom}_{\mathbb{K}}(S_d, \mathbb{K})$ as

$$\begin{aligned} \psi^* : S_d &\to \mathbb{K} \\ p &\mapsto \langle \psi, p \rangle_d \end{aligned}$$

Example: $((\xi_0 x_0 + \cdots + \xi_n x_n)^d)^* = \mathfrak{e}_\xi : p \in S_d \mapsto p(\xi)$ (evaluation at ξ)

Dual symmetric tensor decomposition problem:

Given $\psi^* \in S_d^*$, find a decomposition of the form $\psi^* = \sum_{i=1}^r \omega_i \mathfrak{e}_{\xi_i}$ where $\xi_i = (\xi_{i,0}, \xi_{i,1}, \ldots, \xi_{i,n})$ span distinct lines in $\overline{\mathbb{K}}^{n+1}$, $\omega_i \in \overline{\mathbb{K}}$ $(\omega_i \neq 0)$.

Symmetric tensor decomposition

$$
\begin{aligned}
\psi &= (\mathbf{x_0} + 3\,\mathbf{x_1} - \mathbf{x_2})^4 + (\mathbf{x_0} + \mathbf{x_1} + \mathbf{x_2})^4 - 3\,(\mathbf{x_0} + 2\,\mathbf{x_1} + 2\,\mathbf{x_2})^4 \\
&= -x_0{}^4 - 24\,x_0{}^3 x_2 - 8\,x_0{}^3 x_1 - 60\,x_0{}^2 x_2{}^2 - 168\,x_0{}^2 x_1 x_2 - 12\,x_0{}^2 x_1{}^2 \\
&\quad -96\,x_0 x_2{}^3 - 240\,x_0 x_1 x_2^2 - 384\,x_0 x_1^2 x_2 + 16\,x_0 x_1{}^3 - 46\,x_2{}^4 - 200\,x_1 x_2^3 \\
&\quad -228\,x_1{}^2 x_2{}^2 - 296\,x_1^3 x_2 + 34\,x_1{}^4 \\
\psi^* &\equiv \mathfrak{e}_{(3,-1)}(\mathbf{y}) + \mathfrak{e}_{(1,1)}(\mathbf{y}) - 3\mathfrak{e}_{(2,2)}(\mathbf{y}) \qquad (\textit{by apolarity for } \psi^* : p \mapsto \langle \psi, p \rangle_d)
\end{aligned}
$$

$$
H^{\mathbf{2},\mathbf{2}}_{\psi^*} := \begin{bmatrix}
-1 & -2 & -6 & -2 & -14 & -10 \\
-2 & -2 & -14 & 4 & -32 & -20 \\
-6 & -14 & -10 & -32 & -20 & -24 \\
-2 & 4 & -32 & 34 & -74 & -38 \\
-14 & -32 & -20 & -74 & -38 & -50 \\
-10 & -20 & -24 & -38 & -50 & -46
\end{bmatrix}
$$

For $B = \{1, x_{\mathbf{1}}, x_{\mathbf{2}}\}$,

$$
H^{B,B}_{\psi^*} = \begin{bmatrix} -1 & -2 & -6 \\ -2 & -2 & -14 \\ -6 & -14 & -10 \end{bmatrix}
$$

$$
H^{B,x_{\mathbf{1}}B}_{\psi^*} = \begin{bmatrix} -2 & -2 & -14 \\ -2 & 4 & -32 \\ -14 & -32 & -20 \end{bmatrix}
$$

$$
H^{B,x_{\mathbf{2}}B}_{\psi^*} = \begin{bmatrix} -6 & -14 & -10 \\ -14 & -32 & -20 \\ -10 & -20 & -24 \end{bmatrix}
$$

- The matrix of multiplication by x_2 in $B = \{1, x_1, x_2\}$ is

$$M_2 = (H_{\psi^*}^{B,B})^{-1} H_{\psi^*}^{B,x_2 B} = \begin{bmatrix} 0 & -2 & -2 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 1 & \frac{5}{2} & \frac{3}{2} \end{bmatrix}.$$

- Its eigenvalues are $[-1, 1, 2]$ and the eigenvectors:

$$U := \begin{bmatrix} 0 & -2 & -1 \\ \frac{1}{2} & \frac{3}{4} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

that is the polynomials

$$U(x) = \left[\ \frac{1}{2}\,x_1 - \frac{1}{2}\,x_2 \quad -2 + \frac{3}{4}\,x_1 + \frac{1}{4}\,x_2 \quad -1 + \frac{1}{2}\,x_1 + \frac{1}{2}\,x_2\ \right].$$

- We deduce the weights and the frequencies:

$$H_{\psi^*}^{[1,x_1,x_2],U} = \begin{bmatrix} 1 & 1 & -3 \\ 1 \times 3 & 1 \times 1 & -3 \times 2 \\ 1 \times -1 & 1 \times 1 & -3 \times 2 \end{bmatrix}$$

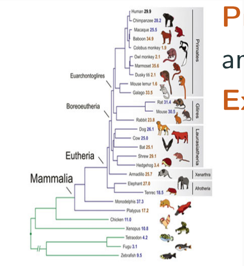
Weights: $1, 1, -3$;
Frequencies: $(-1, 3), (1, 1), (2, 2)$.

Decomposition:

$$\psi^*(\mathbf{y}) = \mathfrak{e}_{(3,-1)}(\mathbf{y}) + \mathfrak{e}_{(1,1)}(\mathbf{y}) - 3\,\mathfrak{e}_{(2,2)}(\mathbf{y}) + \mathcal{O}(\mathbf{y})^4$$

$$\psi(\mathbf{x}) = (\mathbf{x}_0 + 3\,\mathbf{x}_1 - \mathbf{x}_2)^4 + (\mathbf{x}_0 + \mathbf{x}_1 + \mathbf{x}_2)^4 - 3\,(\mathbf{x}_0 + 2\,\mathbf{x}_1 + 2\,\mathbf{x}_2)^4$$

Phylogenetic trees



Problem: study probability vectors for genes $[A, C, G, T]$ and the transitions described by Markov matrices M^i.

Example:

$$\begin{array}{rccc} \textit{Ancestor}: & & \mathcal{A} & \\ \textit{Transitions}: & M^1 & M^2 & M^3 \\ \textit{Species}: & \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 \end{array}$$

For $i_1, i_2, i_3 \in \{A, C, G, T\}$, the probability to observe i_1, i_2, i_3 is

$$p_{i_1,i_2,i_3} = \sum_{k=1}^{4} \pi_k \, M^1_{k,i_1} M^2_{k,i_2} M^3_{k,i_3} \Leftrightarrow \mathbf{p} = \sum_{\mathbf{k=1}}^{\mathbf{4}} \boldsymbol{\pi}_\mathbf{k} \, \mathbf{u}_\mathbf{k} \otimes \mathbf{v}_\mathbf{k} \otimes \mathbf{w}_\mathbf{k}$$

where $\mathbf{u}_k = (M^1_{k,1}, \ldots, M^1_{k,4}), \mathbf{v}_k = (M^2_{k,1}, \ldots, M^2_{k,4}), \mathbf{w}_k = (M^3_{k,1}, \ldots, M^3_{k,4})$.

- p is a tensor $\in \mathbb{K}^4 \otimes \mathbb{K}^4 \otimes \mathbb{K}^4$ of rank ≤ 4.
- Its decomposition yields the M^i and the ancestor probability (π_j).

Multilinear tensor decomposition

A tensor in $\mathbb{K}^4 \otimes \mathbb{K}^4 \otimes \mathbb{K}^4$:

$$\tau := 4\,a_0\,b_0\,c_0 + 7\,a_1\,b_0\,c_0 + 8\,a_2\,b_0\,c_0 + 9\,a_3\,b_0\,c_0 + 5\,a_0\,b_1\,c_0 - 2\,a_0\,b_2\,c_0 + 11\,a_0\,b_3\,c_0 + 6\,a_0\,b_0\,c_1 + 8\,c_2 + 6\,a_0\,b_0\,c_3 + 21\,a_1\,b_1\,c_0 + 28\,a_2\,b_1\,c_0 + 11\,a_3\,b_1\,c_0 - 14\,a_1\,b_2\,c_0 - 21\,a_2\,b_2\,c_0 - 10\,a_3\,b_2\,c_0 + 48\,a_1\,b_3\,c_0 + 65\,a_2\,b_3\,c_0 + 28\,a_3\,b_3\,c_0 + 26\,a_1\,b_0\,c_1 + 35\,a_2\,b_0\,c_1 + 14\,a_3\,b_0\,c_1 + 18\,a_0\,b_1\,c_1 - 10\,a_0\,b_2\,c_1 + 40\,a_0\,b_3\,c_1 + 36\,a_1\,b_0\,c_2 + 48\,a_2\,b_0\,c_2 + 18\,a_3\,b_0\,c_2 + 26\,a_0\,b_1\,c_2 - 9\,a_0\,b_2\,c_2 + 55\,a_0\,b_3\,c_2 + 38\,a_1\,b_0\,c_3 + 53\,a_2\,b_0\,c_3 + 14\,a_3\,b_0\,c_3 + 26\,a_0\,b_1\,c_3 - 16\,a_0\,b_2\,c_3 + 58\,a_0\,b_3\,c_3 + 68\,a_1\,b_1\,c_1 + 91\,a_2\,b_1\,c_1 + 48\,a_3\,b_1\,c_1 - 72\,a_1\,b_2\,c_1 - 105\,a_2\,b_2\,c_1 - 36\,a_3\,b_2\,c_1 + 172\,a_1\,b_3\,c_1 + 235\,a_2\,b_3\,c_1 + 112\,a_3\,b_3\,c_1 + 90\,a_1\,b_1\,c_2 + 118\,a_2\,b_1\,c_2 + 68\,a_3\,b_1\,c_2 - 85\,a_1\,b_2\,c_2 - 127\,a_2\,b_2\,c_2 - 37\,a_3\,b_2\,c_2 + 223\,a_1\,b_3\,c_2 + 301\,a_2\,b_3\,c_2 + 151\,a_3\,b_3\,c_2 + 96\,a_1\,b_1\,c_3 + 129\,a_2\,b_1\,c_3 + 72\,a_3\,b_1\,c_3 - 114\,a_1\,b_2\,c_3 - 165\,a_2\,b_2\,c_3 - 54\,a_3\,b_2\,c_3 + 250\,a_1\,b_3\,c_3 + 343\,a_2\,b_3\,c_3 + 166\,a_3\,b_3\,c_3.$$

Take $a_0 = b_0 = c_0 = 1$. For $B := (1, a_1, a_2, a_3)$ and $B' := (1, b_1, b_2, b_3)$, the corresponding matrix $\mathbb{H}_{\tau^*}^{B,B'}$

$$\mathbb{H}_{\tau^*}^{B,B'} = \begin{pmatrix} 4 & 7 & 8 & 9 \\ 5 & 21 & 28 & 11 \\ -2 & -14 & -21 & -10 \\ 11 & 48 & 65 & 28 \end{pmatrix}$$

is invertible. The transposed operators of multiplication by the variables c_1, c_2, c_3 are:

$$^t\mathbb{M}_{c_{\mathbf{1}}}^{B} = \begin{pmatrix} 0 & 11/6 & -2/3 & -1/6 \\ -2 & -41/6 & 20/3 & 19/6 \\ -2 & -85/6 & 37/3 & 29/6 \\ -2 & 5/2 & 0 & 1/2 \end{pmatrix}$$

$$^t\mathbb{M}_{c_{\mathbf{2}}}^{B} = \begin{pmatrix} -2 & 23/3 & -13/3 & -1/3 \\ -6 & 1/3 & 7/3 & 13/3 \\ -6 & -28/3 & 29/3 & 20/3 \\ -6 & 14 & -7 & 0 \end{pmatrix}$$

$$^t\mathbb{M}_{c_{\mathbf{3}}}^{B} = \begin{pmatrix} 0 & 3/2 & 0 & -1/2 \\ -2 & -33/2 & 14 & 11/2 \\ -2 & -57/2 & 23 & 17/2 \\ -2 & 3/2 & 2 & -1/2 \end{pmatrix}$$

The eigenvalues are respectively $(-1,-2,-3)$, $(2,4,2)$, $(4,5,6)$, and $(1,1,1)$. The corresponding common eigenvectors are:

$$v_{\mathbf{1}} = \begin{pmatrix} 1 \\ -1 \\ -2 \\ 3 \end{pmatrix}, v_{\mathbf{2}} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 2 \end{pmatrix}, v_{\mathbf{3}} = \begin{pmatrix} 1 \\ 5 \\ 7 \\ 3 \end{pmatrix}, v_{\mathbf{4}} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

We deduce that the coordinates $(a_1, a_2, a_3, c_1, c_2, c_3)$ of the 4 points $\xi_1, \ldots, \xi_4$.

Computing the eigenvectors of the operators of multiplications ${}^t\mathbb{M}_{c_1}^{B'}, {}^t\mathbb{M}_{c_2}^{B'}, {}^t\mathbb{M}_{c_3}^{B'}$ we get the coordinates b_1, b_2, b_3 and deduce the 4 points of the decomposition:

$$\xi_{\mathbf{1}} = \begin{pmatrix} -1 \\ -2 \\ 3 \\ -1 \\ -1 \\ -1 \\ -1 \\ -2 \\ -3 \end{pmatrix}, \xi_{\mathbf{2}} := \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 3 \\ 2 \\ 4 \\ 2 \end{pmatrix}, \xi_{\mathbf{3}} = \begin{pmatrix} 5 \\ 7 \\ 3 \\ 3 \\ -4 \\ 8 \\ 4 \\ 5 \\ 6 \end{pmatrix}, \xi_{\mathbf{4}} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Finally, we solve the following linear system in $(\omega_1, \omega_2, \omega_3, \omega_4)$:

$$
\begin{aligned}
T &= \omega_1\,(1 - a_1 - 2\,a_2 + 3\,a_3)\,(1 - b_1 - b_2 - b_3)\,(1 - c_1 - 2\,c_2 - 3\,c_3) \\
&+ \omega_3\,(1 + 2\,a_1 + 2\,a_2 + 2\,a_3)\,(1 + 2\,b_1 + 2\,b_2 + 3\,b_3)\,(1 + 2\,c_1 + 4\,c_2 + 2\,c_3) \\
&+ \omega_3\,(1 + 5\,a_1 + 7\,a_2 + 3\,a_3)\,(1 + 3\,b_1 - 4\,b_2 + 8\,b_3)\,(1 + 4\,c_1 + 5\,c_2 + 6\,c_3), \\
&+ \omega_4\,(1 + a_1 + a_2 + a_3)\,(1 + b_1 + b_2 + b_3)\,(1 + c_1 + c_2 + c_3)
\end{aligned}
$$

we get $\omega_1 = \omega_2 = \omega_3 = \omega_4 = 1$.

Basis construction

Computation of a (orthogonal) basis of $\mathcal{A}_\sigma$

Definition: For $p, q \in E$, let $\langle p, q\rangle_\sigma = \langle \sigma \mid p\, q\rangle$.

Projection: For $\mathbf{p}, \mathbf{q} \subset \mathbb{K}[\mathbf{x}]$, $f \in \mathbb{K}[\mathbf{x}]$,

$$proj(f, \mathbf{p}, \mathbf{q}) =: g \ \text{ s.t. } \ f - g \in \langle \mathbf{p}\rangle, g \perp_\sigma \langle \mathbf{q}\rangle$$

Reduction: For $f = \sum_\alpha f_\alpha \mathbf{x}^\alpha \in \mathbb{K}[\mathbf{x}]$ and $\mathbf{k} = \{k_\delta\}_{\delta \in \mathbf{d}}$ with $k_\delta = \mathbf{x}^\delta + \cdots \in \mathbb{K}[\mathbf{x}]$,

$$red(f, \mathbf{k}) =: f - \sum_{\delta \in D} f_\delta k_\delta.$$

For $B = \{\mathbf{x}^{\beta_1}, \ldots, \mathbf{x}^{\beta_r}\}$ suppose we have (p_i, q_i) such that

- $p_i = \mathbf{x}^{\beta_i} + \sum_{j<i} p_{i,j} \mathbf{x}^{\beta_j}$
- $\langle p_i, q_j \rangle_\sigma = \delta_{i,j}$

For a new monomial $\mathbf{x}^\alpha$,

- project it with respect to B: $r_\alpha = proj(\mathbf{x}^\alpha, p, q)$
- check discrepancy:
    - $\langle \mathbf{x}^\gamma, r_\alpha \rangle_\sigma \neq 0$ extend $\mathbf{p}$ with $p_{r+1} = r^\alpha$, $q_{r+1} = \mathbf{x}^\beta$;
    - otherwise add r_α to the set of relations.

Border basis computation

Input: σ_α for $\alpha \in \mathbf{a}$ s.t. $\mathrm{rank} H_\sigma < \infty$.

- Let $\mathbf{b} = \{\,\}$; $\mathbf{c} = \{\,\}$; $\mathbf{d} = \{\,\}$; $\mathbf{k} = \{\,\}$; $\mathbf{n} = \{0\}$; $\mathbf{s} = \mathbf{a}$; $\mathbf{t} = \mathbf{a}$;
- While $\mathbf{n} \neq \emptyset$ do
  - $\tilde{\mathbf{b}} = \mathbf{b}$;
  - For each $\alpha \in \mathbf{n}$,
    1. if $\alpha = 0$ then $p_\alpha = \tilde{p}_\alpha = 1$;
       else
       $p_\alpha = proj(red_K(x_i p_\beta, \{p_\gamma\}_{\gamma \in \mathbf{b}}, \{m_\gamma\}_{\gamma \in \mathbf{b}})$ for $\beta \in \tilde{\mathbf{b}}$ s.t. $\mathbf{x}^\alpha = x_i \mathbf{x}^\beta$;
    2. find the first $\gamma \in \mathbf{t}$ such that $\langle p_\alpha, \mathbf{x}^\gamma \rangle_\sigma \neq 0$;
    3. If such an γ exists then
       $m_\gamma = \frac{1}{\langle p_\alpha, \mathbf{x}^\gamma \rangle_\sigma} \mathbf{x}^\gamma$; $q_\alpha = proj(m_\gamma, \{q_\beta\}_{\beta \in \mathbf{b}}, \{p_\beta\}_{\beta \in \mathbf{b}})$;
       add α to $\mathbf{b}$, p_α to $\mathbf{p}$, γ to $\mathbf{c}$; remove α from $\mathbf{s}$, γ from $\mathbf{t}$;
       else
       add α to $\mathbf{d}$, p_α to $\mathbf{k}$; remove α from $\mathbf{s}$;
  - $\mathbf{n} = \mathtt{next}(\mathbf{b}, \mathbf{d}, \mathbf{s})$;

Output:

- monomial sets $\mathbf{b} = \{\mathbf{x}^{\beta_1}, \ldots, \mathbf{x}^{\beta_r}\}$, $\mathbf{c} = \{\mathbf{x}^{\gamma_1}, \ldots, \mathbf{x}^{\gamma_r}\}$,
- bases $\mathbf{p} = \{p_{\beta_i}\}$, $\mathbf{q} = \{q_{\beta_i}\}$,
- relations $\mathbf{k} = \{p_\alpha\}_{\alpha \in \mathbf{d}}$.

Proposition

Assume $\mathbf{a}$ *is connected to* 1. *If* $\mathbf{d} = \partial \mathbf{b}$, *then there exits* $\tilde{\sigma} \in \mathbb{K}[[\mathbf{y}]]$ *s.t.*

- $\mathrm{rank} H_{\tilde{\sigma}} = r$,
- $(\mathbf{p}, \mathbf{q})$ *are bases of* $\mathcal{A}_{\tilde{\sigma}}$ *pairwise orthogonal for* $\langle \cdot, \cdot \rangle_{\sigma}$,
- $\mathbf{k}$ *is a border basis of* $I_{\tilde{\sigma}}$ *with respect to* B.

Complexity: $\mathcal{O}(r(r+\delta)s)$ where $r = |\mathbf{b}|$, $\delta = |\partial \mathbf{b}|$ $s = |\mathbf{a}|$ $(\delta \leq n\, r)$.

Berlekamp-Massey-Sakata algorithm: *Compute a non-reduced Grobner basis of the recurrence relations valid up to a monomial m.*

$\mathcal{O}(s'(r+\delta)\, s + r\, s'(r+\delta))$ where s' is the maximal number of non-zero terms in the polynomials of the Grobner basis $(r \leq s' \leq s)$.

Remark: If the new monomials $(\in N)$ are chosen according to a monomial ordering $\prec$, then $\mathbf{c} = \mathbf{b}$.

Remark: If $\mathbb{K} = \mathbb{R}$ and $\forall f \in \mathbb{R}[\mathbf{x}]$, $\langle \sigma | f^2 \rangle \geq 0$ then $\mathbf{p} = \mathbf{q}$ is a basis of orthogonal polynomials of $\mathcal{A}_{\sigma}$.

Polynomial interpolation of points

Given a set of points $\Xi = \{\xi_1, \ldots, \xi_r\} \subset \mathbb{C}^n$, we take the **moments**

$$\sigma_\alpha = \sum_{i=1}^{r} \lambda_i \, \xi_i^\alpha$$

for some $\lambda_i \in \mathbb{C} \setminus \{0\}$ and let $\sigma = \sum_{\alpha \in \mathbb{N}^n} \sigma_\alpha \frac{\mathbf{y}^\alpha}{\alpha!}$ be the generating series.

- $I_\sigma = H_\sigma = \mathcal{I}(\xi_1, \ldots, \xi_r)$ **vanishing ideal** of the points;
- I_σ generated by $\ker H_\sigma^{B', B^+}$ for any bases B, B' of $\mathcal{A}_\sigma$ connected to 1;
- The eigenvectors of the operators $M_i = (H_\sigma^{B', B})^{-1} H_\sigma^{B', x_i B}$ are up to a scalar **interpolation polynomials** at the points $\xi_1, \ldots, \xi_r$.

Example:

Take $\xi := \{(0,0),(1,0),(-1,0),(0,1),(0,-1)\}$ and $\sigma_\alpha = \sum_{i=1}^{5} \xi_i^\alpha$ for $|\alpha| \leqslant 6$:

$$\sigma(\mathbf{z}) = 5 + 2z_1^2 + 2z_2^2 + 2z_1^4 + 2z_2^4 + 2z_1^6 + 2z_2^6 + \cdots$$

Basis by orthogonalization:

- $\mathbf{b}_0 = \mathbf{p}_0 = \{1\}$;
- $\mathbf{n}_1 = \{x_1, x_2\}, \mathbf{b}_1 = \mathbf{p}_1 = \{1, x_1, x_2\}$;
- $\mathbf{n}_2 = \{x_1^2, x_1x_2, x_2^2\}$, $\mathbf{b}_2 = \mathbf{b}_1 \cup \{x_1^2, x_2^2\}$,
  $\mathbf{p}_2 = \left\{1, x_1, x_2, x_1^2 - \frac{2}{5}, x_2^2 - \frac{2}{5}\right\}$, $\mathbf{k}_2 = \{x_1x_2\}$
- $\mathbf{n}_3 = \{x_1^3, x_1^2x_2, x_1x_2^2, x_2^3\}$, $\mathbf{b}_3 = \mathbf{b}_2$, $\mathbf{p}_3 = \mathbf{p}_2$,
  $\mathbf{k}_3 = \langle b - \sum_{i=1}^{5} \frac{\langle b, b_k\rangle_\sigma}{\langle b_k, b_k\rangle_\sigma} b_k, b \in \partial \mathbf{b}_2\rangle = \langle x_1^3 - x_1, x_1^2x_2, x_1x_2, x_1x_2^2, x_2^3 - x_2\rangle$.

☞ **Vanishing ideal:** $I_\sigma = (x_1^3 - x_1, x_1x_2, x_2^3 - x_2)$.

☞ **Lagrange basis:**

$1 - {x_1}^2 - {x_2}^2, \frac{1}{2}x_1 + \frac{1}{2}{x_1}^2, -\frac{1}{2}x_1 + \frac{1}{2}{x_1}^2, \frac{1}{2}x_2 + \frac{1}{2}{x_2}^2, -\frac{1}{2}x_2 + \frac{1}{2}{x_2}^2$

Some references (more references in the notes)

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Thanks for your attention

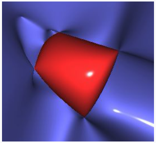
Questions ?

POEMA

Polynomial Optimization, Efficiency through Moments and Algebra

Marie Skłodowska-Curie Innovative Training Network

2019-2022



POEMA network goal is to train scientists at the interplay of algebra, geometry and computer science for polynomial optimization problems and to foster scientific and technological advances, stimulating interdisciplinary and intersectoriality knowledge exchange between algebraists, geometers, computer scientists and industrial actors facing real-life optimization problems.

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10. Univ. of Tromsoe, Norway (Cordian Riener)
11. Artelys SA, Paris, France (Arnaud Renaud)

Associate partners:

1. IBM Research, Ireland (Martin Mevissen)
2. NAG, UK (Mike Dewar)
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