Un cas pratique de la théorie de Picard-Vessiot des équations différentielles non commutatives

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NONCOMMUTATIVE DIFFERENTIAL EQUATION

Notations

- Let $(X^*, 1_{X^*})$ and $(Y^*, 1_{Y^*})$ be the free monoids generated by the alphabets $X = \{x_0, \ldots, x_m\}$ and $Y = \{y_k\}_{k \geq 1}$, respectively.
- $\mathcal{L}ynX$ (resp. $\mathcal{L}ynY$) denotes the set of Lyndon words over X (resp. Y), with $x_0 \prec x_1 \prec \ldots \prec x_m$ (resp. $y_1 \succ y_2 \ldots \succ \ldots$).
- Let Ω be a simply connected domain and $\mathcal{H}(\Omega)$ be the algebra of holomorphic functions on Ω (admitting $1_{\mathcal{H}(\Omega)}$ as neutral element).
- The set of formal power (resp. Lie) series, over X and with coefficients in $\mathcal{H}(\Omega)$, is denoted by $\mathcal{H}(\Omega)\langle\langle X\rangle\rangle$ (resp. $\mathcal{L}ie_{\mathcal{H}(\Omega)}\langle\langle X\rangle\rangle$).
- The differentiation on $\mathcal{H}(\Omega)$ is denoted by ∂_z, *i.e.*
$$\forall c \in \mathcal{H}(\Omega), \quad \partial_z c = 0 \iff c \in \mathbb{C}1_{\mathcal{H}(\Omega)}.$$
- The differentiation on $\mathcal{H}(\Omega)\langle\langle X\rangle\rangle$ is denoted by $\mathbf{d}$, *i.e.*
$$\forall S \in \mathcal{H}(\Omega)\langle\langle X\rangle\rangle, \quad \mathbf{d}S = \sum_{w \in X^*} (\partial_z \langle S|w\rangle) w = 0 \iff S \in \mathbb{C}1_{\mathcal{H}(\Omega)}\langle\langle X\rangle\rangle.$$
- For any $y_i, y_j \in Y$ and $u, v \in Y^*$, one defines[1]
$$u \sqcup\!\sqcup 1_{X^*} = 1_{X^*} \sqcup\!\sqcup u = u, \quad xu \sqcup\!\sqcup yv = x(u \sqcup\!\sqcup yv) + y(xu \sqcup\!\sqcup v),$$
$$u \uplus 1_{Y^*} = 1_{Y^*} \uplus u = u, \quad y_i u \uplus y_j v = y_i(u \uplus y_j v) + y_j(y_i u \uplus v) + y_{i+j}(u \uplus v).$$

1. $\Delta_{\sqcup\!\sqcup}(x) = x \otimes 1_{X^*} + 1_{X^*} \otimes x$, $\Delta_{\uplus}(y_i) = y_i \otimes 1_{Y^*} + 1_{Y^*} \otimes y_i + \sum_{k+l=i} y_k \otimes y_l$

Iterated integrals and Chen series

For $i = 0, \ldots, m$, let $u_i \in \mathcal{C} \subset \mathcal{H}(\Omega)$. The iterated integral associated to $x_{i_1} \ldots x_{i_k} \in X^*$, over the differential forms $\omega_i(z) = u_i(z)dz, i = 0, \ldots, m$, and along a path $z_0 \rightsquigarrow z$ on Ω, is defined by $(\alpha_{z_0}^z(1_{X^*}) = 1_{\mathcal{H}(\Omega)})$

$$\begin{aligned} \alpha_{z_0}^z(x_{i_1} \ldots x_{i_k}) &= \int_{z_0}^z \omega_{i_1}(z_1) \ldots \int_{z_0}^{z_{k-1}} \omega_{i_k}(z_k). \\ \partial_z \alpha_{z_0}^z(x_{i_1} \ldots x_{i_k}) &= u_{i_1}(z) \int_{z_0}^z \omega_{i_2}(z_2) \ldots \int_{z_0}^{z_{k-1}} \omega_{i_k}(z_k). \end{aligned}$$

These iterated integrals satisfy the **Chen**'s lemma, *i.e.*

$$\forall u, v \in X^*, \quad \alpha_{z_0}^z(u \sqcup\!\sqcup v) = \alpha_{z_0}^z(u)\alpha_{z_0}^z(v).$$

The Chen series, over $\omega_0, \ldots, \omega_m$ and along $z_0 \rightsquigarrow z$ on Ω, is defined by

$$C_{z_0 \rightsquigarrow z} := 1_\Omega 1_{X^*} + \sum_{w \in X^*X} \alpha_{z_0}^z(w) w$$

and satisfies the following first order (noncommutative) differential equation

$$(DE) \qquad \mathbf{d}S = MS \quad \text{with} \quad M = u_0 x_0 \ldots + u_m x_m \in \mathcal{C}X \subsetneq \mathcal{L}ie_{\mathcal{C}}\langle X \rangle.$$

By **Ree**'s theorem, $C_{z_0 \rightsquigarrow z} = e^{L_{z_0 \rightsquigarrow z}}$ with $L_{z_0 \rightsquigarrow z} \in \mathcal{L}ie_{\mathcal{C}}\langle\langle X \rangle\rangle \subset \mathcal{C}\langle\langle X \rangle\rangle$.

$(\Delta_{\sqcup\!\sqcup}(C_{z_0 \rightsquigarrow z}) = C_{z_0 \rightsquigarrow z} \otimes C_{z_0 \rightsquigarrow z}$ and $\Delta_{\sqcup\!\sqcup}(M) = 1_{X^*} \otimes M + M \otimes 1_{X^*})$.

$$\mathrm{Gal}(DE) = \{e^C\}_{C \in \mathcal{L}ie_{\mathbb{C}1_\Omega}\langle\langle X \rangle\rangle} \quad \text{and} \quad C_{z_0 \rightsquigarrow z} = \prod_{l \in \mathcal{L}ynX}^{\searrow} e^{\alpha_{z_0}^z(S_l) P_l},$$

where $\{P_l\}_{l \in \mathcal{L}ynX}$ is a basis of Lie algebra $\mathcal{L}ie_{\mathbb{C}}\langle X \rangle$ and $\{S_l\}_{l \in \mathcal{L}ynX}$ is a pure transcendence basis of $(\mathbb{C}\langle X \rangle, \sqcup\!\sqcup, 1_{X^*})$.

Linear and algebraic independence via words

Theorem (Deneufchâtel, Duchamp, HNM & Solomon, 2011, weak and concrete form)

Let $(\mathcal{C}, \partial_z) \subset (\mathcal{H}(\Omega), \partial_z)$ *function field. Let* $S \in \mathcal{H}(\Omega)\langle\langle X \rangle\rangle$ *be a group-like solution of* (DE)*.*
Then the following assertions are equivalent :

1. *the family* $\{\langle S|l\rangle\}_{l \in \mathcal{L}ynX}$ *is algebraic independant over* $(\mathcal{C}, \partial_z)$*.*
2. *the family* $\{\langle S|w\rangle\}_{w \in X^*}$ *is linearly independant over* $(\mathcal{C}, \partial_z)$*.*
3. *the family* $\{\langle S|x\rangle\}_{x \in X \cup \{1_{X^*}\}}$ *is linearly independant over* $(\mathcal{C}, \partial_z)$*.*
4. *the family* $\{u_i\}_{i=0,\ldots,m}$ *of* $\mathcal{C}$ *is such that, for* $c_i \in \mathbb{C}, i = 0, \ldots, m$*, and* $f \in \mathcal{C}$*, one has*
$$c_0 u_0 + \ldots + c_m u_m = \partial_z(f) \implies (\forall i = 1, \ldots, m)(c_i = 0).$$

Example (hyperlogarithms)

$\sigma = \{0 = a_0, a_1, \ldots, a_m\}$ (the a_i's, $i = 0, \ldots, m$, are distinct), $\Omega = \widetilde{\mathbb{C} \setminus \sigma}$,
$\mathcal{C} = \mathbb{C}\{z^{e_0}, (a_1 - z)^{e_1}, \ldots, (a_m - z)^{e_m}\}_{e_0, \ldots, e_m \in \mathbb{C}}$.

$$\mathbf{d}S = MS \quad \text{with} \quad M = \frac{x_0}{z} + \frac{x_1}{a_1 - z} + \ldots + \frac{x_m}{a_m - z}.$$

The case of polylogarithms ($X = \{x_0, x_1\}$, $Y = \{y_k\}_{k\geq 1}$)

$$\Omega = \widetilde{\mathbb{C} \setminus \{0,1\}}, \omega_0(z) = \frac{dz}{z}, \omega_1(z) = \frac{dz}{1-z}, \mathcal{C} = \mathbb{C}\{z^a, (1-z)^b\}_{a,b\in\mathbb{C}}.$$

In this case, $C_{z_0 \rightsquigarrow z} = \mathrm{L}(z)(\mathrm{L}(z_0))^{-1}$, where

$$\mathrm{L} = \sum_{w\in X^*} \mathrm{Li}_w\, w = \prod_{l\in\mathcal{L}ynX}^{\searrow} e^{\mathrm{Li}_{S_l}\, P_l},$$

where, for $n, n_1, \ldots n_r \in \mathbb{N}_+$ and $z \in \mathbb{C}, |\, z\,| < 1$, $\mathrm{Li}_{x_0^n}(z) = \log^n(z)/n!$ and

$$\mathrm{Li}_{x_0^{n_1-1}x_1\ldots x_0^{n_r-1}x_1}(z) = \alpha_0^z(x_0^{n_1-1}x_1 \ldots x_0^{n_r-1}x_1) = \sum_{k_1>\ldots>k_r>0} \frac{z^{k_1}}{k_1^{n_1}\ldots k_1^{n_r}}.$$

The coefficients $\{\mathrm{H}_{y_{s_1}\ldots y_{s_r}}(n)\}_{n\geq 1}$ are defined by the following Taylor expansion

$$\frac{1}{1-z}\mathrm{Li}_{x_0^{n_1-1}x_1\ldots x_0^{n_r-1}x_1}(z) = \sum_{n\geq 0} \mathrm{H}_{y_{s_1}\ldots y_{s_r}}(n) z^n.$$

By a Abel's theorem, for $n_1 > 1$, one has then

$$\zeta(n_1, \ldots, n_r) := \lim_{z\to 1} \mathrm{Li}_{x_0^{n_1-1}x_1\ldots x_0^{n_r-1}x_1}(z) = \lim_{n\to+\infty} \mathrm{H}_{y_{n_1}\ldots y_{n_r}}(n).$$

$$\mathcal{Z} := \mathrm{span}_{\mathbb{Q}}\{\mathrm{Li}_w(1)\}_{w\in x_0X^*x_1} = \mathrm{span}_{\mathbb{Q}}\{\mathrm{H}_w(+\infty)\}_{w\in Y^*\setminus y_1Y^*},$$

using the one-to-one correspondences

$$(s_1, \ldots, s_r) \in \mathbb{N}_+^r \leftrightarrow y_{s_1}\ldots y_{s_r} \in Y^* \underset{\pi_Y}{\overset{\pi_X}{\rightleftharpoons}} x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1 \in X^*x_1.$$

Indexing polylogarithms and harmonic sums by polynomials

The following morphisms are injective

$$\begin{array}{rcl}
\mathrm{Li}_\bullet : (\mathbb{Q}\langle X\rangle, ⧢, 1_{X^*}) & \longrightarrow & (\mathbb{Q}\{\mathrm{Li}_w\}_{w\in X^*}, ., 1), \\
x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1 & \longmapsto & \mathrm{Li}_{x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1} = \mathrm{Li}_{s_1,\ldots,s_r}, \\
\mathrm{H}_\bullet : (\mathbb{Q}\langle Y\rangle, \text{⧢}\!\!\!\!+, 1_{Y^*}) & \longrightarrow & (\mathbb{Q}\{\mathrm{H}_w\}_{w\in Y^*}, ., 1), \\
y_{s_1}\ldots y_{s_r} & \longmapsto & \mathrm{H}_{y_{s_1}\ldots y_{s_r}} = \mathrm{H}_{s_1,\ldots,s_r}.
\end{array}$$

Hence, $\{\mathrm{Li}_l\}_{l\in\mathcal{L}ynX}$ and $\{\mathrm{H}_l\}_{l\in\mathcal{L}ynY}$ are algebraically independent.

The following poly-morphism is, by definition, surjective

$$\zeta : \begin{array}{c}(\mathbb{Q}1_{X^*} \oplus x_0\mathbb{Q}\langle X\rangle x_1, ⧢, 1_{X^*}) \\ (\mathbb{Q}1_{Y^*} \oplus (Y-\{y_1\})\mathbb{Q}\langle Y\rangle, \text{⧢}\!\!\!\!+, 1_{Y^*})\end{array} \twoheadrightarrow (\mathcal{Z}, ., 1),$$

$$\begin{array}{c} x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1 \\ y_{s_1}\ldots y_{s_r}\end{array} \longmapsto \zeta(s_1,\ldots,s_r).$$

It can be extended as characters as follows

$$\begin{array}{rcl}
\zeta_{⧢} : (\mathbb{R}\langle X\rangle, ⧢, 1_{X^*}) & \longrightarrow & (\mathbb{R}, ., 1), \\
\zeta_{\text{⧢}\!\!\!\!+}, \gamma_\bullet : (\mathbb{R}\langle Y\rangle, \text{⧢}\!\!\!\!+, 1_{Y^*}) & \longrightarrow & (\mathbb{R}, ., 1),
\end{array}$$

s.t. $\zeta_{⧢}(x_0) = 0 = \log(1)$, $\zeta_{\text{⧢}\!\!\!\!+}(l) = \zeta_{\text{⧢}\!\!\!\!+}(\pi_Y l) = \gamma_{\pi_Y l} = \zeta(l)$, $(l \in \mathcal{L}ynX - X)$.

$$\begin{array}{rclll}
\zeta_{⧢}(x_1) = & 0 & = \mathrm{f.p.}_{z\to1}\log(1-z), & \{(1-z)^a\log^b(1-z)\}_{a\in\mathbb{Z},b\in\mathbb{N}}, \\
\zeta_{\text{⧢}\!\!\!\!+}(y_1) = & 0 & = \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_1(n), & \{n^a\mathrm{H}_1^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}, \\
\gamma_{y_1} = & \gamma & = \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_1(n), & \{n^a\log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}.
\end{array}$$

In all the sequel, let $\mathcal{X}$ denotes X or Y and $\mathbb{C}\langle\langle\mathcal{X}\rangle\rangle$ denotes the set formal power series, over $\mathcal{X}$ and with coefficients in $\mathbb{C}$.

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It can be extended as characters as follows

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s.t. $\zeta_ш(x_0) = 0 = \log(1)$, $\zeta_⧢(l) = \zeta_⧢(\pi_Y l) = \gamma_{\pi_Y l} = \zeta(l)$, $(l \in \mathcal{L}ynX - X)$.

$$\begin{array}{rcll} \zeta_ш(x_1) = & 0 & = \mathrm{f.p.}_{z\to 1}\log(1-z), & \{(1-z)^a\log^b(1-z)\}_{a\in\mathbb{Z},b\in\mathbb{N}}, \\ \zeta_⧢(y_1) = & 0 & = \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_1(n), & \{n^a\mathrm{H}_1^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}, \\ \gamma_{y_1} = & \gamma & = \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_1(n), & \{n^a\log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}. \end{array}$$

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s.t. $\zeta_{⧢}(x_0) = 0 = \log(1)$, $\zeta_{⧢\!\!\!\!+}(l) = \zeta_{⧢\!\!\!\!+}(\pi_Y l) = \gamma_{\pi_Y l} = \zeta(l)$, $(l \in \mathcal{L}ynX - X)$.

$$\begin{array}{rclll} \zeta_{⧢}(x_1) &=& 0 &= \mathrm{f.p.}_{z\to1}\log(1-z), & \{(1-z)^a\log^b(1-z)\}_{a\in\mathbb{Z},b\in\mathbb{N}}, \\ \zeta_{⧢\!\!\!\!+}(y_1) &=& 0 &= \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_1(n), & \{n^a\mathrm{H}_1^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}, \\ \gamma_{y_1} &=& \gamma &= \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_1(n), & \{n^a\log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}. \end{array}$$

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Indexing polylogarithms by noncommutative rational series

Noncommutative multivariate exponential transforms ($x_0x_1 \neq x_1x_0$) :

$$\begin{array}{cccccc} x_0^n & \longmapsto & \log^n(z)/n!, & x_1^n & \longmapsto & \log^n((1-z)^{-1})/n!, \\ (tx_0)^* & \longmapsto & z^t, & (tx_1)^* & \longmapsto & (1-z)^{-t}. \end{array}$$

Example (polylogarithms indexed by rational series)

$$\mathrm{Li}_{x_0^*}(z) = z, \quad \mathrm{Li}_{x_1^*}(z) = (1-z)^{-1}, \quad \mathrm{Li}_{(ax_0+bx_1)^*}(z) = z^a(1-z)^{-b}.$$

Let $w = y_{s_1}\ldots y_{s_r} \in Y^*$ and $R_w \in (\mathbb{Z}[x_1^*], ш, 1_{X^*})$ by

$$R_w = \sum_{k_1=0}^{s_1} \ldots \sum_{k_r=0}^{\substack{(s_1+\ldots+s_r)-\\(k_1+\ldots+k_{r-1})}} \binom{s_1}{k_1} \ldots \binom{\sum_{i=1}^{r} s_i - \sum_{i=1}^{r-1} k_i}{k_r} \rho_{k_1} ш \ldots ш \rho_{k_r},$$

where, for any $i = 1, \ldots, r$, if $k_i = 0$ then $\rho_{k_i} = x_1^* - 1_{X^*}$ else

$$\rho_{k_i} = x_1^* ш \sum_{j=1}^{k_i} S_2(k_i, j) j! (x_1^* - 1_{X^*})^{ш j}$$

and the $S_2(k_i, j)$'s are the Stirling numbers of second kind. Then

$$\mathrm{Li}_{R_{y_{s_1}\ldots y_{s_r}}}(z) = \mathrm{Li}_{-s_1,\ldots,-s_r}(z) := \sum_{k_1>\ldots>k_r>0} k_1^{s_1}\ldots k_1^{s_r} z^{k_1}.$$

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where, for any $i = 1, \ldots, r$, if $k_i = 0$ then $\rho_{k_i} = x_1^* - 1_{X^*}$ else

$$\rho_{k_i} = x_1^* ⧢ \sum_{j=1}^{k_i} S_2(k_i, j) j! (x_1^* - 1_{X^*})^{⧢ j}$$

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$$\mathrm{Li}_{R_{y_{s_1}\ldots y_{s_r}}}(z) = \mathrm{Li}_{-s_1,\ldots,-s_r}(z) := \sum_{k_1 > \ldots > k_r > 0} k_1^{s_1} \ldots k_1^{s_r} z^{k_1}.$$

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$$(tx_0)^* \longmapsto z^t, \quad (tx_1)^* \longmapsto (1-z)^{-t}.$$

Example (polylogarithms indexed by rational series)

$$\mathrm{Li}_{x_0^*}(z) = z, \quad \mathrm{Li}_{x_1^*}(z) = (1-z)^{-1}, \quad \mathrm{Li}_{(ax_0+bx_1)^*}(z) = z^a(1-z)^{-b}.$$

Let $w = y_{s_1} \ldots y_{s_r} \in Y^*$ and $R_w \in (\mathbb{Z}[x_1^*], ⧢, 1_{X^*})$ by

$$R_w = \sum_{k_1=0}^{s_1} \ldots \sum_{k_r=0}^{\substack{(s_1+\ldots+s_r)-\\(k_1+\ldots+k_{r-1})}} \binom{s_1}{k_1} \ldots \binom{\sum_{i=1}^r s_i - \sum_{i=1}^{r-1} k_i}{k_r} \rho_{k_1} ⧢ \ldots ⧢ \rho_{k_r},$$

where, for any $i = 1, \ldots, r$, if $k_i = 0$ then $\rho_{k_i} = x_1^* - 1_{X^*}$ else

$$\rho_{k_i} = x_1^* ⧢ \sum_{j=1}^{k_i} S_2(k_i, j) j! (x_1^* - 1_{X^*})^{⧢ j}$$

and the $S_2(k_i, j)$'s are the Stirling numbers of second kind. Then

$$\mathrm{Li}_{R_{y_{s_1}\ldots y_{s_r}}}(z) = \mathrm{Li}_{-s_1,\ldots,-s_r}(z) := \sum_{k_1>\ldots>k_r>0} k_1^{s_1} \ldots k_1^{s_r} z^{k_1}.$$

REPRESENTATIVE SERIES

Representative series and $\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$ ($\mathcal{X} = X$ or Y)

Let $\mathbb{C}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ and $\mathbb{C}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$ denote the sets of noncommutative rational and exchangeable[2], respectively, series over $\mathcal{X}$.

1. $(\mathbb{C}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle, ⧢, 1_{\mathcal{X}^*}, \Delta_{\mathrm{conc}}, \mathrm{e}) = (\mathbb{C}\langle\mathcal{X}\rangle, \mathrm{conc}, \Delta_{⧢}, 1_{\mathcal{X}^*}, \mathrm{e})^{\circ}$.
2. The x^*'s, $x \in \mathcal{X}$, are group-like, for Δ_{conc}, and are algebraically independent over $(\mathbb{C}\langle\mathcal{X}\rangle, ⧢, 1_{\mathcal{X}^*})$ within $(\mathbb{C}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle, ⧢, 1_{\mathcal{X}^*})$. So are y^*'s, $y \in Y^*$, over $(\mathbb{C}\langle Y\rangle, ⧢, 1_{Y^*})$ within $(\mathbb{C}^{\mathrm{rat}}\langle\langle Y\rangle\rangle, ⧢, 1_{Y^*})$.
3. $\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle := \mathbb{C}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle \bigcap \mathbb{C}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle = ⧢\{\mathbb{C}^{\mathrm{rat}}\langle\langle x\rangle\rangle\}_{x\in\mathcal{X}}$ and $\forall x \in \mathcal{X}, \mathbb{C}^{\mathrm{rat}}\langle\langle x\rangle\rangle = \mathrm{span}_{\mathbb{C}}\{(ax)^* ⧢ \mathbb{C}\langle x\rangle | a \in \mathbb{C}\}$.
4. $R \in \mathbb{C}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ iff it admits a representation, (ν, μ, η), of dimension n : $\nu \in M_{1,n}(\mathbb{C}), \eta \in M_{n,1}(\mathbb{C}), \mu : \mathcal{X}^* \to M_{n,n}(\mathbb{C})$ s.t.
$$R = \sum_{w\in\mathcal{X}^*} (\nu\mu(w)\eta)w = \nu\Big(\sum_{x\in\mathcal{X}} \mu(x)x\Big)^*\eta.$$
5. Let (ν, μ, η) be of minimal dimension of $R \in \mathbb{C}\langle\langle\mathcal{X}\rangle\rangle$ and $\mathcal{L}$ be the Lie algebra generated by $\{\mu(x)\}_{x\in\mathcal{X}}$. Then
$$R \in \mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle \text{ iff } \mathcal{L} \text{ is commutative.}$$

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2. *i.e.* if $S \in \mathbb{C}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$ then $(\forall u, v \in \mathcal{X}^*)((\forall x \in \mathcal{X})(|u|_x = |v|_x) \Rightarrow \langle S|u\rangle = \langle S|v\rangle$.

Representative series and $\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$ ($\mathcal{X} = X$ or Y)

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Representative series and $\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$ ($\mathcal{X} = X$ or Y)

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Linear representations and automata

For $i = 1, 2$, let $R_i \in \mathbb{C}^{\mathrm{rat}}\langle\langle \mathcal{X} \rangle\rangle$ and (ν_i, μ_i, η_i) be, respectively, representations of dimension n_i. Then the linear representation of

$$\begin{aligned}
R_1 + R_2 \quad &\text{is} \quad \left(\begin{pmatrix} \nu_1 & \nu_2 \end{pmatrix}, \left\{ \begin{pmatrix} \mu_1(x) & \mathbf{0} \\ \mathbf{0} & \mu_2(x) \end{pmatrix} \right\}_{x \in \mathcal{X}}, \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \right), \\
R_1 ⧢ R_2 \quad &\text{is} \quad (\nu_1 \otimes \nu_2, \{\mu_1(x) \otimes \mathrm{I}_{n_2} + \mathrm{I}_{n_1} \otimes \mu_2(x)\}_{x \in \mathcal{X}}, \eta_1 \otimes \eta_2), \\
R_1 \uplus R_2 \quad &\text{is} \quad (\nu_1 \otimes \nu_2, \{\mu_1(y_k) \otimes \mathrm{I}_{n_2} + \mathrm{I}_{n_1} \otimes \mu_2(y_k) + \textstyle\sum_{i+j=k} \\
&\qquad \mu_1(y_i) \otimes \mu_2(y_j)\}_{k \geq 1}, \eta_1 \otimes \eta_2).
\end{aligned}$$

Example (of $(-t^2 x_0 x_1)^*$ and $(t^2 x_0 x_1)^*$)

$(-t^2 x_0 x_1)^*$

$(t^2 x_0 x_1)^*$

$$\nu_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \eta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mu_1(x_0) = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, \quad \mu_1(x_1) = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix},$$

$$\nu_2 = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mu_2(x_0) = \begin{pmatrix} 0 & it \\ 0 & 0 \end{pmatrix}, \quad \mu_2(x_1) = \begin{pmatrix} 0 & 0 \\ it & 0 \end{pmatrix}$$

$$\begin{aligned}
(\nu, \{\mu(x_0), \mu(x_1)\}, \eta) \quad = \quad & (\nu_1 \otimes \nu_2, \{\mu_1(x_0) \otimes \mathrm{I}_{n_2} + \mathrm{I}_{n_1} \otimes \mu_2(x_0), \\
& \mu_1(x_1) \otimes \mathrm{I}_{n_2} + \mathrm{I}_{n_1} \otimes \mu_2(x_1)\}, \eta_1 \otimes \eta_2)
\end{aligned}$$

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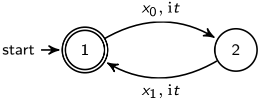
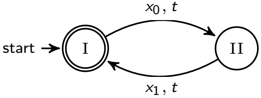
$$R_1 + R_2 \quad \text{is} \quad \left(\begin{pmatrix} \nu_1 & \nu_2 \end{pmatrix}, \left\{ \begin{pmatrix} \mu_1(x) & \mathbf{0} \\ \mathbf{0} & \mu_2(x) \end{pmatrix} \right\}_{x \in \mathcal{X}}, \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \right),$$

$$R_1 ⧢ R_2 \quad \text{is} \quad (\nu_1 \otimes \nu_2, \{\mu_1(x) \otimes \mathrm{I}_{n_2} + \mathrm{I}_{n_1} \otimes \mu_2(x)\}_{x \in \mathcal{X}}, \eta_1 \otimes \eta_2),$$

$$R_1 \uplus R_2 \quad \text{is} \quad (\nu_1 \otimes \nu_2, \{\mu_1(y_k) \otimes \mathrm{I}_{n_2} + \mathrm{I}_{n_1} \otimes \mu_2(y_k) + \textstyle\sum_{i+j=k} \mu_1(y_i) \otimes \mu_2(y_j)\}_{k \geq 1}, \eta_1 \otimes \eta_2).$$

Example (of $(-t^2x_0x_1)^*$ and $(t^2x_0x_1)^*$)





$(-t^2x_0x_1)^*$ $\qquad$ $(t^2x_0x_1)^*$

$$\nu_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \eta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mu_1(x_0) = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, \quad \mu_1(x_1) = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix},$$

$$\nu_2 = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mu_2(x_0) = \begin{pmatrix} 0 & \mathrm{i}t \\ 0 & 0 \end{pmatrix}, \quad \mu_2(x_1) = \begin{pmatrix} 0 & 0 \\ \mathrm{i}t & 0 \end{pmatrix}$$

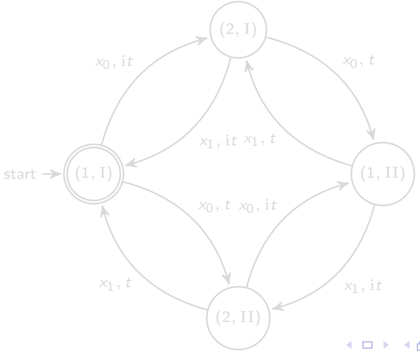
$$(\nu, \{\mu(x_0), \mu(x_1)\}, \eta) = (\nu_1 \otimes \nu_2, \{\mu_1(x_0) \otimes \mathrm{I}_{n_2} + \mathrm{I}_{n_1} \otimes \mu_2(x_0), \mu_1(x_1) \otimes \mathrm{I}_{n_2} + \mathrm{I}_{n_1} \otimes \mu_2(x_1)\}, \eta_1 \otimes \eta_2).$$

Example of $(-t^2x_0x_1)^* \sqcup\!\sqcup (t^2x_0x_1)^* = (-4t^4x_0^2x_1^2)^*$

$$\nu = \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}, \quad \eta = {}^T\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix},$$

$$\mu(x_0) = \begin{pmatrix}0 & 0 & t & 0\\ 0 & 0 & 0 & t\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & \mathrm{i}t & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \mathrm{i}t\\ 0 & 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & \mathrm{i}t & t & 0\\ 0 & 0 & 0 & t\\ 0 & 0 & 0 & \mathrm{i}t\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$\mu(x_1) = \begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ t & 0 & 0 & 0\\ 0 & t & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0 & 0 & 0\\ \mathrm{i}t & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & \mathrm{i}t & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & 0\\ \mathrm{i}t & 0 & 0 & 0\\ t & 0 & 0 & 0\\ 0 & t & \mathrm{i}t & 0\end{pmatrix}.$$



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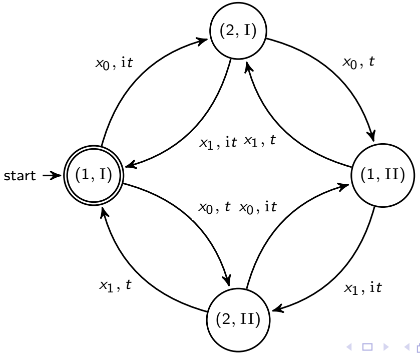
$$\mu(x_0) = \begin{pmatrix}0 & 0 & t & 0\\ 0 & 0 & 0 & t\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & it & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & it\\ 0 & 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & it & t & 0\\ 0 & 0 & 0 & t\\ 0 & 0 & 0 & it\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$\mu(x_1) = \begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ t & 0 & 0 & 0\\ 0 & t & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0 & 0 & 0\\ it & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & it & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & 0\\ it & 0 & 0 & 0\\ t & 0 & 0 & 0\\ 0 & t & it & 0\end{pmatrix}.$$



Sub bialgebras of $(\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle, ⧢, 1_{X^*}, \Delta_{\mathtt{conc}}, \mathrm{e})$

Let (ν, μ, η) be of minimal dimension of $R \in \mathbb{C}\langle\langle X\rangle\rangle$ and $\mathcal{L}$ be the Lie algebra generated by $\{\mu(x)\}_{x\in X}$.

Letting $M(x) := \mu(x)x$, for $x \in X$, one has $R = \nu M(X^*)\eta$ and

$$M(X^*) = (M(x_1^*)M(x_0))^* M(x_1^*) = (M(x_0^*)M(x_1))^* M(x_0^*).$$

Moreover, if $\{\mu(x)\}_{x\in X}$are triangular then let $D(X)$ (resp. $N(X)$) be diagonal (resp. nilpotent) letter matrix s.t. $M(X) = D(X) + N(X)$. One has

$$M(X^*) = ((D(X^*)T(X))^* D(X^*)).$$

On the other hand, the modules generated by the following families are closed by $\mathtt{conc}$, ⧢ and coproducts :

(F_0) $E_1 x_{i_1} \ldots E_j x_{i_j} E_{j+1}$, where $x_{i_k} \in X, E_k \in \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle$,

(F_1) $E_1 x_{i_1} \ldots E_j x_{i_j} E_{j+1}$, where $x_{i_k} \in X, E_k \in \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle$,

(F_2) $E_1 x_{i_1} \ldots E_j x_{i_j} E_{j+1}$, where $x_{i_k} \in X, E_k \in \mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle$.

It follows then,

1. R is linear combination of expressions in the form (F_0) (resp. (F_1)) iff $M(x_1^*)M(x_0)$ (resp. $M(x_0^*)M(x_1)$) is nilpotent,
2. R is linear combination of expressions in the form (F_2) iff $\mathcal{L}$ is solvable. Hence, if $R \in \mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle ⧢ \mathbb{C}\langle X\rangle$ then $\mathcal{L}$ is solvable.

Sub bialgebras of $(\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle, \shuffle, 1_{X^*}, \Delta_{\mathtt{conc}}, \mathrm{e})$

Let (ν, μ, η) be of minimal dimension of $R \in \mathbb{C}\langle\langle X\rangle\rangle$ and $\mathcal{L}$ be the Lie algebra generated by $\{\mu(x)\}_{x\in X}$.

Letting $M(x) := \mu(x)x$, for $x \in X$, one has $R = \nu M(X^*)\eta$ and

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Moreover, if $\{\mu(x)\}_{x\in X}$are triangular then let $D(X)$ (resp. $N(X)$) be diagonal (resp. nilpotent) letter matrix s.t. $M(X) = D(X) + N(X)$.

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MORE ABOUT CONTANTS OF INTEGRATION

Extension of $\mathrm{Li}_\bullet$ ($\mathcal{C} = \mathbb{C}\{z^a, (1-z)^b\}_{a,b\in\mathbb{C}}$)

Theorem

1. $\{\mathrm{Li}_w\}_{w\in X^*}$ *is* $\mathcal{C}$*-linearly independent. Moreover, the kernel of the following map is the* $\sqcup\!\sqcup$ *-ideal is generated by* $x_0^* \sqcup\!\sqcup\, x_1^* - x_1^* + 1$
$$\mathrm{Li}_\bullet : (\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle \sqcup\!\sqcup\, \mathbb{C}\langle X\rangle, \sqcup\!\sqcup\,, 1_{X^*}) \twoheadrightarrow (\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}, ., 1_\Omega), \quad R \mapsto \mathrm{Li}_R$$
2. *The algebra* $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ *is closed under the differential operators* $\theta_0 = z\partial_z, \theta_1 = (1-z)\partial_z$*, and under their sections*[3] ι_0, ι_1.
3. *The bi-integro differential algebra* $(\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}, \theta_0, \theta_1, \iota_0, \iota_1)$ *is closed under the action of the group of transformations,* $\mathcal{G}$*, generated by* $\{z \mapsto 1-z, z \mapsto z^{-1}\}$*, permuting* $\{0, 1, +\infty\}$ *:*
$$\forall h \in \mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}, \quad \forall g \in \mathcal{G}, \quad h(g) \in \mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}.$$
4. *If* $R \in \mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle \sqcup\!\sqcup\, \mathbb{C}\langle X\rangle$ *(resp.* $\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle$*) then* $\mathrm{Li}_R \in \mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ *(resp.* $\mathcal{C}[\log(z), \log(1-z)]$*).*
5. *If* $R \in \mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ *of minimal representation of dimension* N *then*
$$y(z_0, z) = \alpha_{z_0}^z(R) =: \langle R \parallel C_{z_0\rightsquigarrow z}\rangle = \langle R \parallel \mathrm{L}(z)(\mathrm{L}(z_0))^{-1}\rangle.$$
Moreover, $\{\partial_z^n y\}_{0\le n\le N-1}$ *are* $\mathcal{C}$*-linearly independent and there exists* $a_N, \ldots, a_1, a_0 \in \mathcal{C}$ *such that*
$$(a_N\partial_z^N + a_{N-1}\partial_z^{N-1} + \ldots + a_1\partial_z + a_0)y = 0.$$

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3. *i.e.* $\theta_0\iota_0 = \theta_1\iota_1 = \mathrm{Id}$. Note also that $[\theta_0, \theta_1] = \theta_0 + \theta_1 = \partial_z$.

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Extension of $\mathrm{H}_\bullet$

Lemma (Kleene stars of the plane)

For any $r \geq 1$, the arithmetic function $\mathrm{H}_{y_r^*}$ is transcendent and

$$\forall t \in \mathbb{C}, |\, t \,| < 1, \quad \mathrm{H}_{(t^r y_r)^*} = \sum_{k\geq 0} \mathrm{H}_{y_r^k} t^{kr} = \exp\left(\sum_{k\geq 1} \mathrm{H}_{y_{kr}} \frac{(-t^r)^{k-1}}{k} \right).$$

By identification the coefficients of t^k and by injectivity, one gets

$$y_r^* = \exp_{\uplus}\left(\sum_{k\geq 1} y_{kr} \frac{(-1)^{k-1}}{k} \right),$$

$$y_r^k = \frac{(-1)^k}{k!} \sum_{\substack{s_1,\ldots,s_k>0 \\ s_1+\ldots+ks_k=k}} \frac{(-y_r)^{\uplus s_1}}{1^{s_1}} \uplus \ldots \uplus \frac{(-y_{kr})^{\uplus s_k}}{k^{s_k}}.$$

Lemma

For any $s \geq 1$, let $a_s, b_s \in \mathbb{C}$. Then

$$\left(\sum_{s\geq 1} a_s y_s \right)^* \uplus \left(\sum_{s\geq 1} b_s y_s \right)^* = \left(\sum_{s\geq 1} (a_s + b_s) y_s + \sum_{r,s\geq 1} a_s b_r y_{s+r} \right)^*.$$

Hence, for $|\, a_s \,| < 1, |\, b_s \,| < 1, |\, a_s + b_s \,| < 1,$

$$\mathrm{H}_{(\sum_{s\geq 1}(a_s+b_s)y_s + \sum_{r,s\geq 1} a_s b_r y_{s+r})^*} = \mathrm{H}_{(\sum_{s\geq 1} a_s y_s)^*} \mathrm{H}_{(\sum_{s\geq 1} b_s y_s)^*}.$$

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Hence, for $|a_s| < 1, |b_s| < 1, |a_s + b_s| < 1$,

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Families of eulerian functions

For $r \geq 2$ and $|t| < 1$, let

$$f_1(t) := \gamma t - \sum_{k \geq 2} \zeta(k) \frac{(-t)^k}{k} \text{ and } f_r(t) := \sum_{k \geq 1} \zeta(kr) \frac{(-t^r)^{k-1}}{k}.$$

Proposition

The family $\{f_r\}_{r \geq 1}$ is linearly independent and the family $\{\exp(f_r)\}_{r \geq 1}$ is linearly independent.

For any $r \geq 1$ and $|t| < 1$, one put[4] $\Gamma_{y_r}(1+t) := e^{-f_r(t)}$ s.t.

$$\frac{1}{\Gamma_{y_1}(1+t)} = \exp\Big(\gamma t - \sum_{k \geq 2} \zeta(k) \frac{(-t)^k}{k}\Big) = e^{\gamma t} \prod_{n \geq 1} \Big(1 + \frac{t}{n}\Big) e^{-\frac{t}{n}},$$

$$\frac{1}{\Gamma_{y_r}(1+t)} = \exp\Big(\sum_{k \geq 1} \zeta(kr) \frac{(-t^r)^{k-1}}{k}\Big) = \prod_{n \geq 1} \Big(1 + \frac{t^r}{n^r}\Big),$$

and

$$\mathrm{B}_{y_r}(a, b) := \frac{\Gamma_{y_r}(a)\Gamma_{y_r}(b)}{\Gamma_{y_r}(a+b)}.$$

4. Note that $\Gamma_{y_1}(t) = \Gamma(t)$ and $\mathrm{B}_{y_1}(a, b) = \mathrm{B}(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$.

Families of eulerian functions

For $r \geq 2$ and $|t| < 1$, let

$$f_1(t) := \gamma t - \sum_{k \geq 2} \zeta(k) \frac{(-t)^k}{k} \text{ and } f_r(t) := \sum_{k \geq 1} \zeta(kr) \frac{(-t^r)^{k-1}}{k}.$$

Proposition

The family $\{f_r\}_{r \geq 1}$ is linearly independent and the family $\{\exp(f_r)\}_{r \geq 1}$ is linearly independent.

For any $r \geq 1$ and $|t| < 1$, one put[4] $\Gamma_{y_r}(1+t) := e^{-f_r(t)}$ s.t.

$$\frac{1}{\Gamma_{y_1}(1+t)} = \exp\Big(\gamma t - \sum_{k \geq 2} \zeta(k) \frac{(-t)^k}{k}\Big) = e^{\gamma t} \prod_{n \geq 1} \Big(1 + \frac{t}{n}\Big) e^{-\frac{t}{n}},$$

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Extended double regularization by Newton-Girard formula

Theorem

The characters $\zeta_{\sqcup\!\sqcup}$ *and* $\gamma_\bullet$ *are extended algebraically as follows*

$$\begin{array}{rcl}\zeta_{\sqcup\!\sqcup}:(\mathbb{C}\langle X\rangle \sqcup\!\sqcup\, \mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\!\langle X\rangle\!\rangle, \sqcup\!\sqcup\,, 1_{X^*}) & \longrightarrow & (\mathbb{C}, ., 1),\\ \forall t\in\mathbb{C}, |t|<1, (tx_0)^*, (tx_1)^* & \longmapsto & 1_{\mathbb{C}}.\\ \gamma_\bullet:(\mathbb{C}\langle Y\rangle \uplus \{\mathbb{C}^{\mathrm{rat}}\langle\!\langle y_r\rangle\!\rangle\}_{r\geq 1}, \uplus, 1_{Y^*}) & \longrightarrow & (\mathbb{C}, ., 1),\\ \forall t\in\mathbb{C}, |t|<1, \forall r\geq 1, (t^r y_r)^* & \longmapsto & \Gamma_{y_r}^{-1}(1+t).\end{array}$$

Moreover, the morphism $(\mathbb{C}[\{(y_r)^*\}_{r\geq 1}], \uplus, 1_{Y^*}) \to (\mathbb{C}[\{\exp(f_r)\}_{r\geq 1}], \times, 1)$, *mapping* y_r^* *to* $\Gamma_{y_r}^{-1}$, *is injective and* $\Gamma_{y_{2r}}(1-t) = \Gamma_{y_r}(1+t)\Gamma_{y_r}(1-t)$.

Corollary

For any[5] $s\geq 1$, *let* $a_s, b_s \in \mathbb{C}$, $|a_s|<1$, $|b_s|<1$, $|a_s+b_s|<1$,

$$\gamma_{(\sum_{s\geq1}(a_s+b_s)y_s+\sum_{r,s\geq1}a_sb_ry_{s+r})^*} = \gamma_{(\sum_{s\geq1}a_sy_s)^*}\gamma_{(\sum_{s\geq1}b_sy_s)^*}.$$

Corollary (comparison formula)

For any $z, a, b\in\mathbb{C}$ *such that* $|z|<1$ *and* $\Re a>0, \Re b>0$, *one has*

$$\mathrm{Li}_{x_0[(ax_0)^* \sqcup\!\sqcup ((1-b)x_1)^*]}(z) = \mathrm{Li}_{x_1[((a-1)x_0)^* \sqcup\!\sqcup (-bx_1)^*]}(z) = \mathrm{B}(z;a,b),$$

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5. In particular, $\gamma_{(a_sy_s+a_ry_r+a_sa_ry_{s+r})^*} = \gamma_{(a_sy_s)^*}\gamma_{(a_ry_r)^*}$ and $\gamma_{(-a_s^2y_{2s})^*} = \gamma_{(a_sy_s)^*}\gamma_{(-a_sy_s)^*}$.

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Riemann zeta function and eulerian functions

For $v = -u$ ($|u| < 1$), one gets

$$\frac{1}{\Gamma(1-u)\Gamma(1+u)} = \exp\left(-\sum_{k\geq 1}\zeta(2k)\frac{u^{2k}}{k}\right) = \frac{\sin(u\pi)}{u\pi}.$$

Taking the logarithms and then taking the Taylor expansions, one obtains

$$\begin{aligned}
-\sum_{k\geq 1}\zeta(2k)\frac{u^{2k}}{k} &= \log\left(1+\sum_{n\geq 1}\frac{(ui\pi)^{2n}}{\Gamma(2n)}\right) \\
&= \sum_{l\geq 1}\frac{(-1)^{l-1}}{l}\sum_{k\geq 1}(ui\pi)^{2k}\sum_{\substack{n_1,\ldots,n_l\geq 1\\ n_1+\ldots+n_l=k}}\prod_{i=1}^{l}\frac{1}{\Gamma(2n_i)} \\
&= \sum_{k\geq 1}(ui\pi)^{2k}\sum_{l\geq 1}\frac{(-1)^{l-1}}{l}\sum_{\substack{n_1,\ldots,n_l\geq 1\\ n_1+\ldots+n_l=k}}\prod_{i=1}^{l}\frac{1}{\Gamma(2n_i)}.
\end{aligned}$$

One can deduce then the following expression for $\zeta(2k)$:

$$\frac{\zeta(2k)}{\pi^{2k}} = k\sum_{l=1}^{k}\frac{(-1)^{k+l}}{l}\sum_{\substack{n_1,\ldots,n_l\geq 1\\ n_1+\ldots+n_l=k}}\prod_{i=1}^{l}\frac{1}{\Gamma(2n_i)} \in \mathbb{Q}.$$

Euler gave an other explicit formula using Bernoulli numbers $\{b_k\}_{k\in\mathbb{N}}$:

$$\frac{\zeta(2k)}{(2i\pi)^{2k}} = -\frac{b_{2k}}{2(2k)!} \in \mathbb{Q}.$$

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Polyzetas and extended eulerian functions

$$
\begin{aligned}
& & \gamma_{(-t^2y_2)^*} &= \gamma_{(ty_1)^*}\gamma_{(-ty_1)^*} & & \\
&\Leftrightarrow & \Gamma_{y_2}^{-1}(1-t) &= \Gamma_{y_1}^{-1}(1+t)\Gamma_{y_1}^{-1}(1-t) & & \\
&\Leftrightarrow & e^{-\sum_{k\geq 2}\zeta(2k)t^{2k}/k} &= \frac{\sin(t\pi)}{t\pi} &=& \sum_{k\geq 1}\frac{(ti\pi)^{2k}}{(2k)!}.
\end{aligned}
$$

$$
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& & \gamma_{(-t^4y_4)^*} &= \gamma_{(t^2y_2)^*}\gamma_{(-t^2y_2)^*} & & \\
&\Leftrightarrow & \Gamma_{y_4}^{-1}(1-t) &= \Gamma_{y_2}^{-1}(1+t)\Gamma_{y_2}^{-1}(1-t) & & \\
&\Leftrightarrow & e^{-\sum_{k\geq 1}\zeta(4k)t^{4k}/k} &= \frac{\sin(\mathrm{i}t\pi)}{\mathrm{i}t\pi}\frac{\sin(t\pi)}{t\pi} &=& \sum_{k\geq 1}\frac{2(-4t\pi)^{4k}}{(4k+2)!}.
\end{aligned}
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It follows then, by identification the coeffients of t^{2k} and t^{4k} :

$$\zeta(\overbrace{2,\ldots,2}^{k\text{times}})/\pi^{2k} = 1/(2k+1)! \in \mathbb{Q},$$

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 & \gamma_{(-t^4y_4)^*} & = & \gamma_{(t^2y_2)^*}\gamma_{(-t^2y_2)^*} & \\
\Leftrightarrow & \Gamma_{y_4}^{-1}(1-t) & = & \Gamma_{y_2}^{-1}(1+t)\Gamma_{y_2}^{-1}(1-t) & \\
\Leftrightarrow & e^{-\sum_{k\ge 1}\zeta(4k)t^{4k}/k} & = & \dfrac{\sin(it\pi)}{it\pi}\dfrac{\sin(t\pi)}{t\pi} & = \displaystyle\sum_{k\ge 1}\frac{2(-4t\pi)^{4k}}{(4k+2)!}.
\end{array}$$

Since $\gamma_{(-t^4y_4)^*}=\zeta((-t^4y_4)^*), \gamma_{(-t^2y_2)^*}=\zeta((-t^2y_2)^*), \gamma_{(t^2y_2)^*}=\zeta((t^2y_2)^*)$ then, using the poly-morphism ζ, one deduces

$$\begin{aligned}
\zeta((-t^4y_4)^*) = \zeta((-t^2y_2)^*)\zeta((t^2y_2)^*) &= \zeta((-t^2x_0x_1)^*)\zeta((t^2x_0x_1)^*)) \\
= \zeta((-t^2x_0x_1)^* ⧢ (t^2x_0x_1)^*) &= \zeta((-4t^4x_0^2x_1^2)^*).
\end{aligned}$$

It follows then, by identification the coeffients of t^{2k} and t^{4k} :

$$\zeta(\overbrace{2,\ldots,2}^{k\text{times}})/\pi^{2k} = 1/(2k+1)! \in \mathbb{Q},$$

$$\zeta(\overbrace{3,1,\ldots,3,1}^{k\text{times}})/\pi^{4k} = 4^k\zeta(\overbrace{4,\ldots,4}^{k\text{times}})/\pi^{4k} = 2/(4k+2)! \in \mathbb{Q}.$$

THANK YOU FOR YOUR ATTENTION

Polyzetas and extended eulerian functions

$$\begin{array}{lcccc} & \gamma_{(-t^2y_2)^*} & = & \gamma_{(ty_1)^*}\gamma_{(-ty_1)^*} & \\ \Leftrightarrow & \Gamma_{y_2}^{-1}(1-t) & = & \Gamma_{y_1}^{-1}(1+t)\Gamma_{y_1}^{-1}(1-t) & \\ \Leftrightarrow & e^{-\sum_{k\ge2}\zeta(2k)t^{2k}/k} & = & \dfrac{\sin(t\pi)}{t\pi} & = \displaystyle\sum_{k\ge1}\frac{(ti\pi)^{2k}}{(2k)!}. \end{array}$$

$$\begin{array}{lcccc} & \gamma_{(-t^4y_4)^*} & = & \gamma_{(t^2y_2)^*}\gamma_{(-t^2y_2)^*} & \\ \Leftrightarrow & \Gamma_{y_4}^{-1}(1-t) & = & \Gamma_{y_2}^{-1}(1+t)\Gamma_{y_2}^{-1}(1-t) & \\ \Leftrightarrow & e^{-\sum_{k\ge1}\zeta(4k)t^{4k}/k} & = & \dfrac{\sin(it\pi)}{it\pi}\dfrac{\sin(t\pi)}{t\pi} & = \displaystyle\sum_{k\ge1}\frac{2(-4t\pi)^{4k}}{(4k+2)!}. \end{array}$$

Since $\gamma_{(-t^4y_4)^*} = \zeta((-t^4y_4)^*), \gamma_{(-t^2y_2)^*} = \zeta((-t^2y_2)^*), \gamma_{(t^2y_2)^*} = \zeta((t^2y_2)^*)$ then, using the poly-morphism ζ, one deduces

$$\begin{aligned} \zeta((-t^4y_4)^*) = \zeta((-t^2y_2)^*)\zeta((t^2y_2)^*) &= \zeta((-t^2x_0x_1)^*)\zeta((t^2x_0x_1)^*)) \\ = \zeta((-t^2x_0x_1)^* ⧢ (t^2x_0x_1)^*) &= \zeta((-4t^4x_0^2x_1^2)^*). \end{aligned}$$

It follows then, by identification the coeffients of t^{2k} and t^{4k} :

$$\zeta(\overbrace{2,\ldots,2}^{k\texttt{times}})/\pi^{2k} = 1/(2k+1)! \in \mathbb{Q},$$

$$\zeta(\overbrace{3,1,\ldots,3,1}^{k\texttt{times}})/\pi^{4k} = 4^k\zeta(\overbrace{4,\ldots,4}^{k\texttt{times}})/\pi^{4k} = 2/(4k+2)! \in \mathbb{Q}.$$

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