On Exact Polya, Hilbert-Artin and Putinar's Representations

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Joint work with

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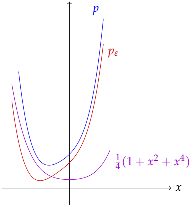
JNCF

04th February 2019



Deciding Non-negativity

$X = (X_1, \ldots, X_n)$ **co-NP hard problem: check $f \geqslant 0$ on $\mathbf{K}$**

$f \in \mathbb{Q}[X]$

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1. Unconstrained $\rightsquigarrow \mathbf{K} = \mathbb{R}^n$

2. Constrained
   $\rightsquigarrow \mathbf{K} = \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geqslant 0, \ldots, g_m(\mathbf{x}) \geqslant 0\} \quad g_j \in \mathbb{Q}[X]$

$\deg f, \deg g_j \leqslant d$

[Collins 75] CAD **doubly exp. in n poly. in d**

[Grigoriev-Vorobjov 88, Basu-Pollack-Roy 98]
Critical points **singly exponential time** $(m+1)\, \tau\, d^{\mathcal{O}(n)}$

Deciding Non-negativity

💡 Sums of squares (SOS)

$\sigma = {h_1}^2 + \cdots + {h_p}^2$

Deciding Non-negativity

Sums of squares (SOS)

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HILBERT 17TH PROBLEM: f SOS of rational functions?

[Artin 27] **YES**!

[Lasserre/Parrilo 01] **Numerical** solvers compute σ

Semidefinite programming (SDP) $\rightsquigarrow$ **approximate** certificates

$$\boxed{\simeq \quad \rightarrow \quad =}$$

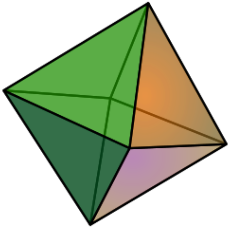
The Question of Exact Certification

How to go from **approximate** to **exact** certification?

What is Semidefinite Programming?

- Linear Programming (LP):

$$\begin{aligned} \min_{\mathbf{z}} \quad & \mathbf{c}^\top \mathbf{z} \\ \text{s.t.} \quad & \mathbf{A}\,\mathbf{z} \geqslant \mathbf{d} \,. \end{aligned}$$



Polyhedron

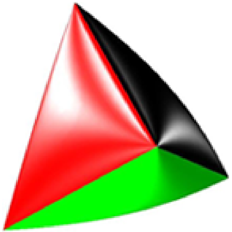
  - Linear cost $\mathbf{c}$
  - Linear inequalities "$\sum_i A_{ij} z_j \geqslant d_i$"

What is Semidefinite Programming?

- Semidefinite Programming (SDP):

$$\min_{\mathbf{z}} \quad \mathbf{c}^\top \mathbf{z}$$
$$\text{s.t.} \quad \sum_i \mathbf{F}_i \, z_i \succcurlyeq \mathbf{F}_0 \ .$$

  - Linear cost $\mathbf{c}$
  - Symmetric matrices $\mathbf{F}_0, \mathbf{F}_i$
  - Linear matrix inequalities "$\mathbf{F} \succcurlyeq 0$" ($\mathbf{F}$ has nonnegative eigenvalues)



Spectrahedron

What is Semidefinite Programming?

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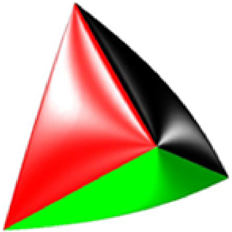
$$\text{s.t.} \quad \sum_i \mathbf{F}_i \, z_i \succcurlyeq \mathbf{F}_0 \;, \quad \mathbf{A}\,\mathbf{z} = \mathbf{d} \;.$$



Spectrahedron

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Lasserre's Hierarchy

- Prove **polynomial inequalities** with SDP:

$$f(a,b) := a^2 - 2ab + b^2 \geqslant 0 \; .$$

- Find $\mathbf{z}$ s.t. $f(a,b) = \begin{pmatrix} a & b \end{pmatrix} \underbrace{\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}}_{\succcurlyeq 0} \begin{pmatrix} a \\ b \end{pmatrix}$.

- Find $\mathbf{z}$ s.t. $a^2 - 2ab + b^2 = z_1 a^2 + 2 z_2 ab + z_3 b^2 \qquad (\mathbf{A}\,\mathbf{z} = \mathbf{d})$

- $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F}_1} z_1 + \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\mathbf{F}_2} z_2 + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{F}_3} z_3 \succcurlyeq \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F}_0}$

Lasserre's Hierarchy

- Choose a cost $\mathbf{c}$ e.g. $(1, 0, 1)$ and solve:

$$\min_{\mathbf{z}} \quad \mathbf{c}^\top \mathbf{z}$$
$$\text{s.t.} \quad \sum_i \mathbf{F}_i \, z_i \succcurlyeq \mathbf{F}_0 \,, \quad \mathbf{A}\,\mathbf{z} = \mathbf{d} \,.$$

- Solution $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \succcurlyeq 0$ (eigenvalues 0 and 2)

- $a^2 - 2ab + b^2 = \begin{pmatrix} a & b \end{pmatrix} \underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}_{\succcurlyeq 0} \begin{pmatrix} a \\ b \end{pmatrix} = (a - b)^2 \,.$

- Solving **SDP** $\implies$ Finding **Sums of Squares** certificates

Lasserre's Hierarchy

NP hard General Problem: $f^* := \min_{\mathbf{x} \in \mathbf{K}} f(\mathbf{x})$

- Semialgebraic set
  $\mathbf{K} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geqslant 0, \ldots, g_m(\mathbf{x}) \geqslant 0\}$

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- $:= [0,1]^2 = \{\mathbf{x} \in \mathbb{R}^2 : x_1(1 - x_1) \geqslant 0, \quad x_2(1 - x_2) \geqslant 0\}$

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$$\overbrace{x_1 x_2}^{f} = -\frac{1}{8} + \overbrace{\frac{1}{2}\left(x_1 + x_2 - \frac{1}{2}\right)^2}^{\sigma_0} + \overbrace{\frac{1}{2}}^{\sigma_1} \overbrace{x_1(1 - x_1)}^{g_1} + \overbrace{\frac{1}{2}}^{\sigma_2} \overbrace{x_2(1 - x_2)}^{g_2}$$

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- Σ = Sums of squares (SOS) σ_i

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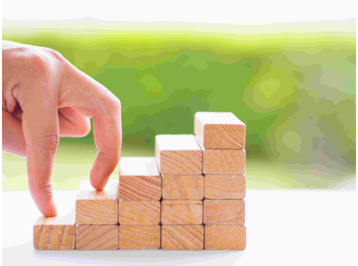
- Bounded degree:
  $\mathcal{Q}_d(\mathbf{K}) := \left\{ \sigma_0 + \sum_{j=1}^{m} \sigma_j g_j, \text{ with } \deg \sigma_j \, g_j \leqslant 2d \right\}$

Lasserre's Hierarchy



- **Hierarchy of SDP relaxations**:
  $\lambda_d := \sup \Big\{ \lambda : f - \lambda \in \mathcal{Q}_d(\mathbf{K}) \Big\}$

- Convergence guarantees $\lambda_d \uparrow f^*$ [Lasserre 01]

- Can be computed with SDP solvers (CSDP, SDPA)

- **"No Free Lunch" Rule**: $\binom{n+2d}{n}$ SDP variables

Certifying Non-negativity

Approximate solutions

sum of squares of $a^2 - 2ab + b^2$?

$(1.00001a - 0.99998b)^2$!

$$a^2 - 2ab + b^2 \simeq (1.00001a - 0.99998b)^2$$

$$a^2 - 2ab + b^2 \neq 1.0000200001a^2 - 1.9999799996ab + 0.9999600004b^2$$

$$\boxed{\simeq \quad \rightarrow \quad =}\ ?$$

Certifying Non-negativity

1. **Polya**'s representation $f = \frac{\sigma}{(X_1^2 + \cdots + X_n^2)^D}$
   positive definite form f
   [Reznick 95]

2. **Hilbert-Artin**'s representation $f = \frac{\sigma}{h^2}$
   $f \geqslant 0$
   [Artin 27]

3. **Putinar**'s representation
   $f = \sigma_0 + \sigma_1 \, g_1 + \cdots + \sigma_m \, g_m$ $f > 0$ on compact K
   $\deg \sigma_i \leqslant 2D$
   [Putinar 93]

One Answer when $\mathbf{K} = \mathbb{R}^n$

Hybrid **SYMBOLIC/NUMERIC** methods

[Peyrl-Parrilo 08] [Kaltofen-Yang-Zhi 08]

$\rightsquigarrow$ can handle degenerate situations when $f \in \partial\Sigma$

$$f(X) \simeq \mathbf{v}_D{}^T(X)\, \tilde{\mathbf{Q}}\, \mathbf{v}_D(X) \quad \tilde{\mathbf{Q}} \succcurlyeq 0$$

$\mathbf{v}_D(X)$: vector of monomials of $\deg \leqslant D$

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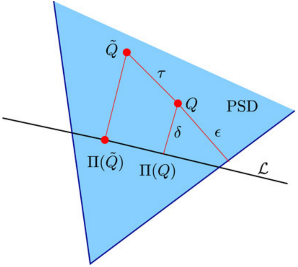

$\tilde{\mathbf{Q}}$ Rounding $\mathbf{Q}$ Projection $\prod(\mathbf{Q})$

$$f(X) = \mathbf{v}_D{}^T(X) \prod(\mathbf{Q})\, \mathbf{v}_D(X)$$

$\prod(\mathbf{Q}) \succcurlyeq 0$ when $\varepsilon \to 0$

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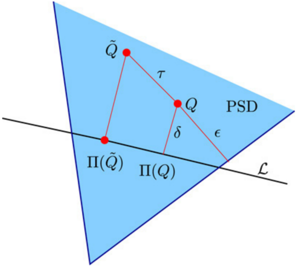

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COMPLEXITY?

One Answer when $\mathbf{K} = \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geqslant 0\}$

Hybrid **SYMBOLIC/NUMERIC** methods

Magron-Allamigeon-Gaubert-Werner 14

$$f \simeq \tilde{\sigma}_0 + \tilde{\sigma}_1\, g_1 + \cdots + \tilde{\sigma}_m\, g_m$$

$$u = f - \tilde{\sigma}_0 + \tilde{\sigma}_1\, g_1 + \cdots + \tilde{\sigma}_m\, g_m$$

One Answer when $\mathbf{K} = \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geqslant 0\}$

Hybrid **SYMBOLIC/NUMERIC** methods

Magron-Allamigeon-Gaubert-Werner 14

$f \simeq \tilde{\sigma}_0 + \tilde{\sigma}_1 \, g_1 + \cdots + \tilde{\sigma}_m \, g_m$

Compact $\mathbf{K} \subseteq [0,1]^n$

$u = f - \tilde{\sigma}_0 + \tilde{\sigma}_1 \, g_1 + \cdots + \tilde{\sigma}_m \, g_m$

$\boxed{\simeq \quad \rightarrow \quad =}$

$\forall \mathbf{x} \in [0,1]^n, u(\mathbf{x}) \leqslant -\varepsilon$

$\min_{\mathbf{K}} f \geqslant \varepsilon$ when $\varepsilon \to 0$

COMPLEXITY?

Related Work: Exact Methods

Existence Question

Does there exist $h_i \in \mathbb{Q}[X], c_i \in \mathbb{Q}^{>0}$ s.t. $f = \sum_i c_i \, {h_i}^2$?

Related Work: Exact Methods

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Does there exist $h_i \in \mathbb{Q}[X], c_i \in \mathbb{Q}^{>0}$ s.t. $f = \sum_i c_i \, {h_i}^2$?

$$\boxed{n = 1 \quad \deg f = d}$$

- $f = c_1 \, {h_1}^2 + c_2 \, {h_2}^2 + c_3 \, {h_3}^2 + c_4 \, {h_4}^2 + c_5 \, {h_5}^2$ [Pourchet 72]
- $f = c_1 \, {h_1}^2 + \cdots + c_d \, {h_d}^2$ [Schweighofer 99]
- $f = c_1 \, {h_1}^2 + \cdots + c_{d+3} \, {h_{d+3}}^2$ [Chevillard et. al 11]

Related Work: Exact Methods

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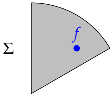
$$n > 1 \quad \deg f = d$$

SOS with Exact LMIs $f = \mathbf{v}_d{}^T(X) \, \mathbf{G} \, \mathbf{v}_d{}^T(X)$ $\mathbf{G} \succcurlyeq 0$

- Solving over the rationals [Guo-Safey El Din-Zhi 13]
- Solving over the reals [Henrion-Naldi-Safey El Din 16]

The Cost of Exact Polynomial Optimization

$f \in \mathbb{Q}[\mathbf{X}] \cap \mathring{\Sigma}[X]$ (interior of the SOS cone)
bit size τ $\quad \deg f = d$



Complexity Question(s)

What is the output bit size of $\sum_i c_i \, h_i{}^2$?

1. **Polya**'s representation $\qquad f = \frac{\sigma}{(X_1^2 + \cdots + X_n^2)^D}$
   positive definite form f
2. **Hilbert-Artin**'s representation $\qquad f = \frac{\sigma}{h^2}$
   $f \geqslant 0$ and $\sigma \in \mathring{\Sigma}[X]$
3. **Putinar**'s representation
   $f = \sigma_0 + \sigma_1 \, g_1 + \cdots + \sigma_m \, g_m$
   $f > 0$ on compact $\mathbf{K}$ $\qquad \deg \sigma_i \leqslant 2D$

Exact algorithm? **Bounds** on D, $\tau(\sigma_i)$?

Contributions

Complexity cost of certifying non-negativity

Algorithm `intsos` $\rightsquigarrow$ **OUTPUT BIT SIZE** $= \tau\, d^{\mathcal{O}(n)}$

Similar complexity cost $d^{\mathcal{O}(n)}$ for **Deciding**

1. **Polya**'s representation

   positive definite form f

   Algorithm `Polyasos`

   OUTPUT BIT SIZE $= 2^{\tau\, d^{\mathcal{O}(n)}}$

2. **Hilbert-Artin**'s representation

   $f = \frac{\sigma}{h^2}$

   $\deg h = D \quad \tau(h) = \tau_D$

   Algorithm `Hilbertsos`

   OUTPUT BIT SIZE $= \tau_D\, D^{\mathcal{O}(n)}$

3. **Putinar**'s representation

   $f > 0$ on compact $\mathbf{K}$

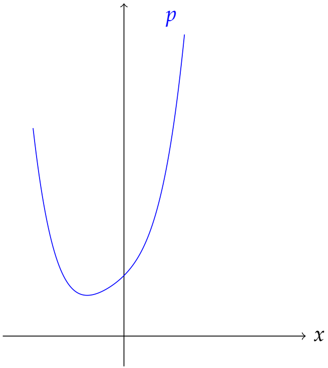
   Algorithm `Putinarsos`

   OUTPUT BIT SIZE $= \mathcal{O}\left(2^{\tau\, d^{n\, C_{\mathbf{K}}}}\right)$

intsos with $n = 1$ and SDP Approximation

Algorithm adapted from [Chevillard-Harrison-Joldes-Lauter 11]

$p \in \mathbb{Z}[X]$, $\deg p = d = 2k$, $p > 0$



$p = 1 + X + X^2 + X^3 + X^4$

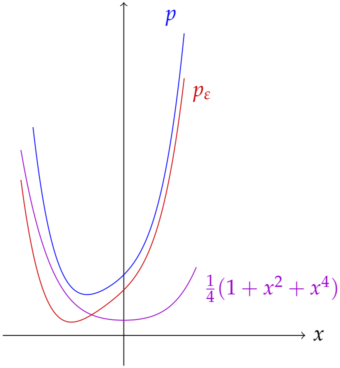
intsos with $n = 1$ and SDP Approximation

Algorithm adapted from [Chevillard-Harrison-Joldes-Lauter 11]

$p \in \mathbb{Z}[X]$, deg $p = d = 2k$, $p > 0$

Perturb: find $\varepsilon \in \mathbb{Q}$ s.t.

$$p_\varepsilon := p - \varepsilon \sum_{i=0}^{k} X^{2i} > 0$$



$$p = 1 + X + X^2 + X^3 + X^4$$

$$\varepsilon = \frac{1}{4}$$

$$p > \frac{1}{4}(1 + X^2 + X^4)$$

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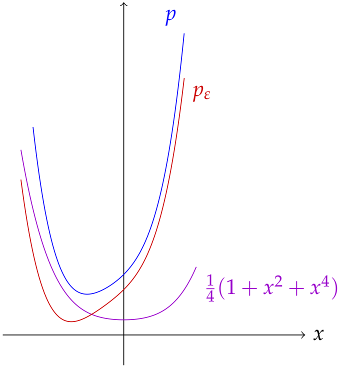
SDP approximation:

$$p - \varepsilon \sum_{i=0}^{k} X^{2i} = {s_1}^2 + {s_2}^2 + u$$

ABSORB: small enough u_i

$\implies \varepsilon \sum_{i=0}^{k} X^{2i} + u$ SOS



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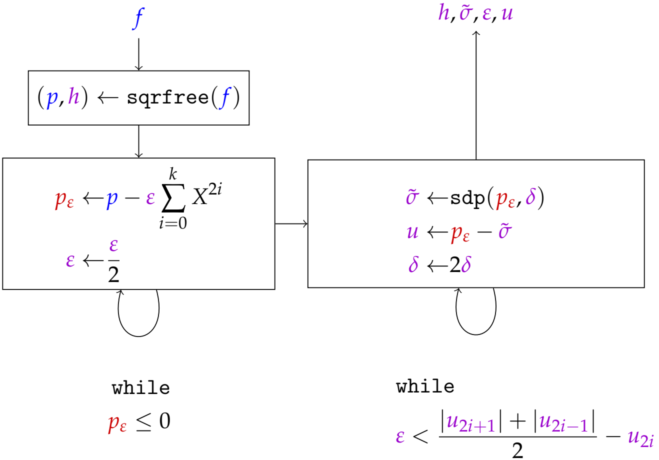
intsos with $n = 1$ and SDP Approximation

- **Input**: $f \geqslant 0 \in \mathbb{Q}[X]$ of degree $d \geqslant 2$, $\varepsilon \in \mathbb{Q}^{>0}$, $\delta \in \mathbb{N}^{>0}$
- **Output**: SOS decomposition with coefficients in $\mathbb{Q}$



intsos with $n = 1$: Absorbtion

- $X = \frac{1}{2}\left[(X+1)^2 - 1 - X^2\right]$
- $-X = \frac{1}{2}\left[(X-1)^2 - 1 - X^2\right]$

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$$u_{2i+1} X^{2i+1} = \frac{|u_{2i+1}|}{2}\left[(X^{i+1} + \operatorname{sgn}(u_{2i+1}) X^i)^2 - X^{2i} - X^{2i+2}\right]$$

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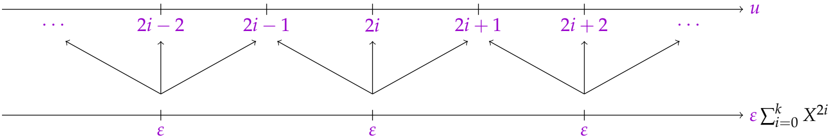

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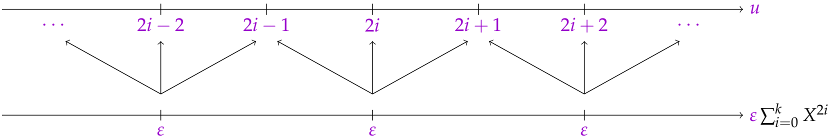
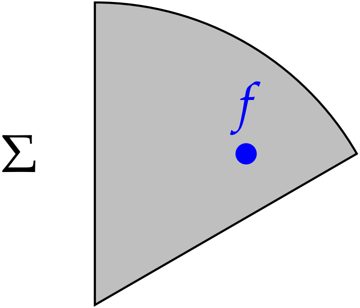
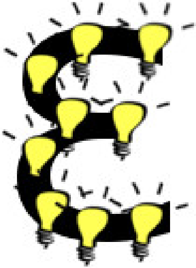


$$\varepsilon \geqslant \frac{|u_{2i+1}| + |u_{2i-1}|}{2} - u_{2i} \implies \varepsilon \sum_{i=0}^{k} X^{2i} + u \quad \text{SOS}$$

intsos with $n \geqslant 1$: Perturbation





PERTURBATION idea

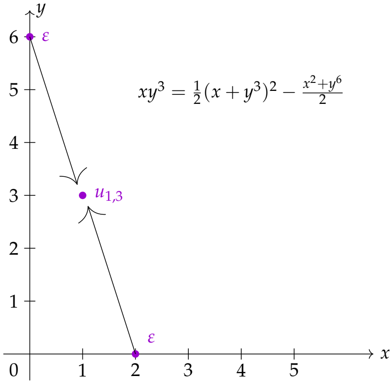
Approximate SOS Decomposition

$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

intsos with $n \geqslant 1$: Absorbtion

$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

Choice of $\mathcal{P}$?



`intsos` with $n \geqslant 1$: Absorbtion

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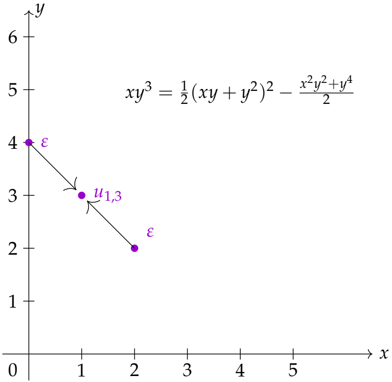

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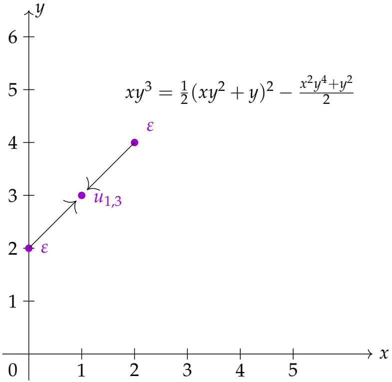


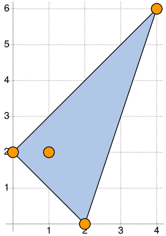
intsos with $n \geqslant 1$: Absorbtion

$$f(X) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} X^{2\alpha} = \tilde{\sigma} + u$$

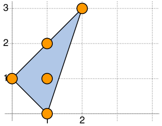
Choice of $\mathcal{P}$?

$f = 4x^4y^6 + x^2 - xy^2 + y^2$
spt(f) = $\{(4,6),(2,0),(1,2),(0,2)\}$

Newton Polytope $\mathcal{P} = \mathrm{conv}\,(\mathrm{spt}(f))$

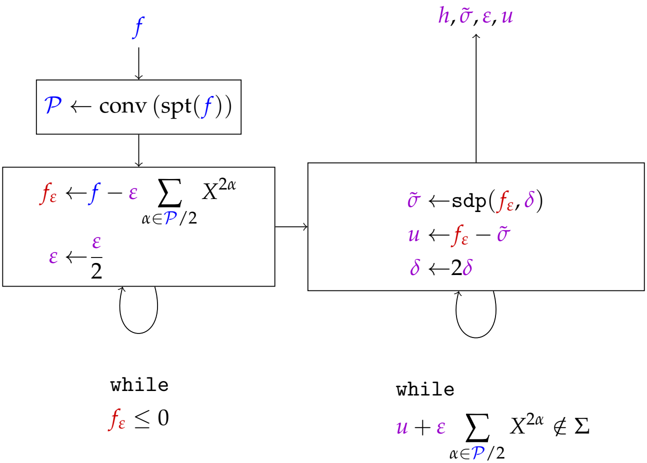


Squares in SOS decomposition $\subseteq \frac{\mathcal{P}}{2} \cap \mathbb{N}^n$
[Reznick 78]



Algorithm `intsos`

- **Input**: $f \in \mathbb{Q}[\mathbf{X}] \cap \mathring{\Sigma}[X]$ of degree d, $\varepsilon \in \mathbb{Q}^{>0}$, $\delta \in \mathbb{N}^{>0}$
- **Output**: SOS decomposition with coefficients in $\mathbb{Q}$



Algorithm `intsos`

Theorem (Exact Certification Cost in $\mathring{\Sigma}$)

$f \in \mathbb{Q}[X] \cap \mathring{\Sigma}[X]$ with $\deg f = d = 2k$ and bit size τ

$\Longrightarrow$ `intsos` terminates with SOS output of bit size $\boxed{\tau \, d^{\mathcal{O}\,(n)}}$

Algorithm intsos

Theorem (Exact Certification Cost in $\mathring{\Sigma}$)

$f \in \mathbb{Q}[X] \cap \mathring{\Sigma}[X]$ with $\deg f = d = 2k$ and bit size τ

$\implies$ `intsos` terminates with SOS output of bit size $\boxed{\tau\, d^{\mathcal{O}(n)}}$

Proof.

- $\{\varepsilon \in \mathbb{R} : \forall \mathbf{x} \in \mathbb{R}^n, f(\mathbf{x}) - \varepsilon \sum_{\alpha \in \mathcal{P}/2} \mathbf{x}^{2\alpha} \geqslant 0\} \neq \emptyset$
Quantitative height & degree bounds for Quantifier Elimination
[Basu-Pollack-Roy 06] $\implies \tau(\varepsilon) = \tau\, d^{\mathcal{O}(n)}$

- # Coefficients in SOS output = size($\mathcal{P}/2$) = $\binom{n+k}{n} \leqslant d^n$

- Ellipsoid algorithm for SDP [Grötschel-Lovász-Schrijver 93] □

Algorithm `Polyasos`

positive definite form f has **Polya**'s representation:

$$f = \frac{\sigma}{(X_1^2 + \cdots + X_n^2)^D} \quad \text{with } \sigma \in \Sigma[X]$$

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Theorem

$$f \cdot (X_1^2 + \cdots + X_n^2)^D \in \Sigma[X] \implies f \cdot (X_1^2 + \cdots + X_n^2)^{D+1} \in \mathring{\Sigma}[X]$$

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Apply Algorithm `intsos` on $f \cdot (X_1^2 + \cdots + X_n^2)^{D+1}$

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Apply Algorithm `intsos` on $f \cdot (X_1^2 + \cdots + X_n^2)^{D+1}$

Theorem (Exact Certification Cost of Polya's representations)

$f \in \mathbb{Q}[X]$ positive definite form with $\deg f = d$ and bit size τ

$$\implies D \leqslant 2^{\tau\, d^{\mathcal{O}(n)}} \qquad \text{OUTPUT BIT SIZE} = \boxed{\tau\, D^{\mathcal{O}(n)} = 2^{\tau\, d^{\mathcal{O}(n)}}}$$

Algorithm `Putinarsos`

$f > 0$ on compact $\mathbf{K} := \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geqslant 0\} \subseteq [-1,1]^n$

Putinar's representation:

$$f = \sigma_0 + \sum_j \sigma_j \, g_j \quad \text{with } \sigma_j \in \Sigma[X] \,, \deg \sigma_j \leqslant 2D$$

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$$f = \mathring{\sigma}_0 + \sum_j \mathring{\sigma}_j \, g_j + \sum_{|\alpha| \leqslant D} c_\alpha (1 - X^{2\alpha})$$

$$\text{with } \mathring{\sigma}_j \in \mathring{\Sigma}[X]\,, \deg \mathring{\sigma}_j \leqslant 2D\,, c_\alpha > 0$$

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ABSORBTION as in Algorithm `intsos`:

$u = f_\varepsilon - \tilde{\sigma}_0 - \sum_j \tilde{\sigma}_j \, g_j - \sum_{|\alpha| \leqslant D} \tilde{c}_\alpha (1 - X^{2\alpha})$

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$$\text{OUTPUT BIT SIZE} = \boxed{\tau \, D^{\mathcal{O}\,(n)} = \mathcal{O}\,(2^{\tau \, d^n \, C_{\mathbf{K}}})}$$

Benchmarks

`RealCertify` library

- Maple 16, Intel Core i7-5600U CPU (2.60 GHz)
- Averaging over five runs

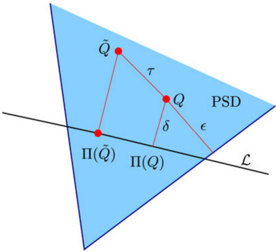
1. Newton Polytope with `convex` Maple package [Franz 99]
2. arbitrary precision SDPA-GMP solver [Nakata 10] $\rightsquigarrow$ `sdp`
3. Cholesky's decomposition with Maple's `LUDecomposition`

Benchmarks: Polya

`RoundProject` [Peyrl-Parrilo 08]

`RAGLib` [Safey El Din] & `CAD` [Moreno Maza]
⇝ exact but no certificate

Bad choice of $\varepsilon, \delta \implies$ `intsos` fails when $f \in \mathring{\Sigma}$



Id	n	d	`multivsos`		`RoundProject`		`RAGLib`	`CAD`
			τ_1 (bits)	t_1 (s)	τ_2 (bits)	t_2 (s)	t_3 (s)	t_4 (s)
f_{20}	2	20	745 419	110.	78 949 497	141.	0.16	0.03
M	3	8	17 232	0.35	18 831	0.29	0.15	0.03
f_2	2	4	1 866	0.03	1 031	0.04	0.09	0.01
f_6	6	4	56 890	0.34	475 359	0.54	623.	–
f_1	10	4	344 347	2.45	8 374 082	4.59	–	–

Benchmarks: Nonnegative polynomials $\notin \mathring{\Sigma}$

S_i from Shapiro conjecture V_i from computational geometry

M_i from monotone permanent conjecture

$\rightsquigarrow$ solved by Kaltofen, Yang & Zhi with SOS certificates

$\implies$ **limitations** of `multivsos`!

Id	n	d	`multivsos`		`RAGLib`	`CAD`
			τ_1 (bits)	t_1 (s)	t_2 (s)	t_3 (s)
S_1	4	24	–	–	1788.	–
S_2	4	24	–	–	1840.	–
V_1	6	8	–	–	5.00	–
V_2	5	18	–	–	1180.	–
M_1	8	8	–	–	351.	–
M_2	8	8	–	–	82.0	–
M_3	8	8	–	–	120.	–
M_4	8	8	–	–	84.0	–

Benchmarks: Putinar

Id	n	d	multivsos k	multivsos τ_1 (bits)	multivsos t_1 (s)	RAGLib t_2 (s)	CAD t_3 (s)
f_{260}	6	3	2	114 642	2.72	4.19	–
f_{491}	6	3	2	108 359	9.65	6.40	0.05
f_{752}	6	2	2	10 204	0.26	0.27	–
f_{859}	6	7	4	6 355 724	303.	0.05	–
f_{863}	4	2	1	5 492	0.14	0.01	0.01
f_{884}	4	4	3	300 784	25.1	113.	–
butcher	6	3	2	247 623	1.32	231.	–
heart	8	4	2	618 847	2.94	24.7	–

Conclusion and Perspectives

Input f on $\mathbf{K}$ with $\deg f = d$ and bit size τ

Algo	Input	$\mathbf{K}$	**OUTPUT BIT SIZE**
`intsos`	$\mathring{\Sigma}$	$\mathbb{R}^n$	$\tau\, d^{\mathcal{O}\,(n)}$
`Hilbertsos`	$\mathring{\Sigma}_D$	$\mathbb{R}^n$	$\tau_D\, D^{\mathcal{O}\,(n)}$
`Polyasos`	pos. def. form	$\mathbb{R}^n$	$2^{\tau\, d^{\mathcal{O}\,(n)}}$
`Putinarsos`	> 0	$\{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geqslant 0\}$	$\mathcal{O}\,(2^{\tau\, d^{n^{C_{\mathbf{K}}}}})$

POLYNOMIAL ALGORITHMS in D = representation degree

Conclusion and Perspectives

Input f on $\mathbf{K}$ with $\deg f = d$ and bit size τ

Algo	Input	$\mathbf{K}$	Output Bit Size
intsos	$\mathring{\Sigma}$	$\mathbb{R}^n$	$\tau\, d^{\mathcal{O}(n)}$
Hilbertsos	$\mathring{\Sigma}_D$	$\mathbb{R}^n$	$\tau_D\, D^{\mathcal{O}(n)}$
Polyasos	pos. def. form	$\mathbb{R}^n$	$2^{\tau\, d^{\mathcal{O}(n)}}$
Putinarsos	> 0	$\{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geqslant 0\}$	$\mathcal{O}\left(2^{\tau\, d^{n^{C_{\mathbf{K}}}}}\right)$

Polynomial algorithms in D = representation degree

- Can give explicit constant $\mathcal{O}(n)$ Improve bounds on D
- Better arbitrary-precision SDP solvers
- When $f \in \partial\Sigma$?
- Extension to other linear/geometric/SDP relaxations

End

Thank you for your attention!

gricad-gitlab:RealCertify

http://www-verimag.imag.fr/~magron

- Magron, Safey El Din & Schweighofer. Algorithms for Weighted Sums of Squares Decomposition of Non-negative Univariate Polynomials, *JSC*. arxiv:1706.03941

- Magron & Safey El Din. On Exact Polya and Putinar's Representations, *ISSAC'18*. arxiv:1802.10339

- Magron & Safey El Din. RealCertify: a Maple package for certifying non-negativity, *ISSAC'18*. arxiv:1805.02201