The lexicographic degree of two-bridge knots

E. Brugallé, P. -V. Koseleff, *D. Pecker*

Sorbonne Université (UPMC – Paris 6), IMJ (UMR CNRS 7586), Ouragan (Inria)

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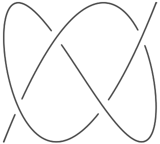
Every knot $K \subset \mathbf{S}^3$ can be represented as the closure of the image of a polynomial embedding $\mathbf{R} \to \mathbf{R}^3 \subset \mathbf{S}^3$, see Vassiliev, 80's.



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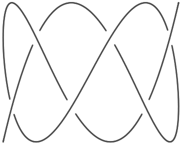

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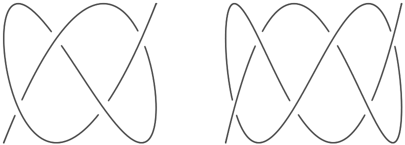

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Definition

The *multidegree* of a polynomial map $\gamma : \mathbf{R} \to \mathbf{R}^n, t \mapsto (P_i(t))$ is the n-tuple $(\deg(P_i))$. The *lexicographic degree* of a knot K is the minimal multidegree, for the lexicographic order, of a polynomial knot whose closure in $\mathbf{S}^3$ is isotopic to K.

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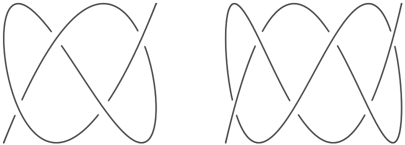

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Aim

Determine the lexicographic degree of a knot K.

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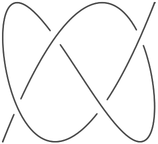

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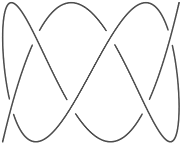

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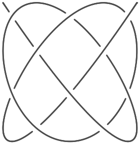
Keywords

Polynomial knots, plane curves, trigonal curves, continued fractions, real pseudoholomorphic curves, knot diagrams, braids



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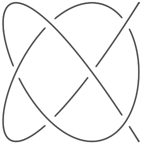

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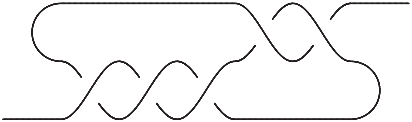
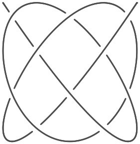


$\bar{5}_2$

Knots $\mathrm{H}(5, 6, 7)$ and $\mathrm{H}(4, 5, 7)$ are isotopic to the twist knot $\bar{5}_2$



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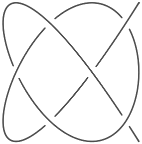

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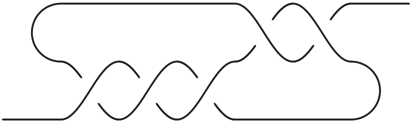

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2-bridge knots admit trigonal diagrams

A two-bridge knot admits a diagram in *Conway's open form* (or trigonal form). This diagram, denoted by $D(m_1, m_2, \ldots, m_k)$ where $m_i \in \mathbf{Z}$

Schubert fraction

The two-bridge links are classified by their Schubert fractions

$$\frac{\alpha}{\beta} = m_1 + \cfrac{1}{m_2 + \cfrac{1}{\cdots + \cfrac{1}{m_k}}} = [m_1, \ldots, m_k], \quad \alpha \geq 0,\ (\alpha, \beta) = 1.$$

$D(m_1, m_2, \ldots, m_k)$ and $D(m'_1, m'_2, \ldots, m'_l)$ correspond to isotopic links if and only if $\alpha = \alpha'$ and $\beta' \equiv \beta^{\pm 1} \pmod{\alpha}$.

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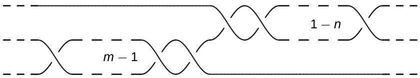
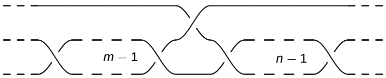
Definition

Let $C(u, m, -n, -v)$ be a trigonal diagram, where m, n are integers, and u, v are (possibly empty) sequences of integers. The Lagrange isotopy twists the right part of the diagram.

$$C(u, m, -n, -v) \mapsto C(u, m - \varepsilon, \varepsilon, n - \varepsilon, v),\ \varepsilon = \pm 1, \tag{1}$$





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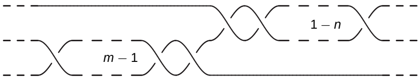

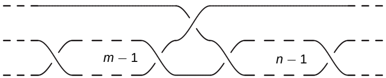

Consequence

Every 2-bridge knot K admits has an alternating diagram of the form $D(m_1, m_2, \ldots m_k)$, where m_i are all positive or all negative. $[u, m, -n, -v] = [u, m - \varepsilon, \varepsilon, n - \varepsilon, v]$

Bounds

Crossing number

The *crossing number* N of K is the minimum number of crossings among all diagrams corresponding to isotopic knots.

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Lower bound (BKP, 2016)

The lexicographic degree of K is more than $(3, N+1, 2N-1)$.
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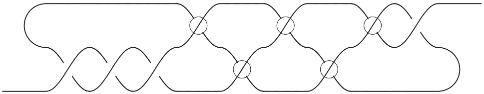
Theorem

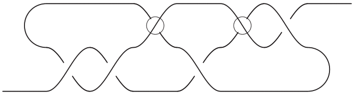
Let $\gamma : \mathbf{R} \to \mathbf{R}^3$ *be a polynomial parametrization of degree* $(3, b, c)$ *of a knot of crossing number* N*. Then we have*

$$b + c \geq 3N.$$

Furthermore, if $N \leq 11$*, then the lexicographic degree of* K *satisfies* $b + c = 3N$*.*

The plane curve C parametrized by $C(t) = (x(t), y(t))$ has $b - 1$ nodes in $\mathbf{C}^2$. Let $N_0 = \sum_{i=1}^{k} |m_i|$ be the number of *real crossings* of C, and let $\delta = b - 1 - N_0$ be the number of other nodes – *solitary nodes* $\in \mathbf{R}^2$, *pairs of complex conjugated nodes in* $\mathbf{C}^2 \setminus \mathbf{R}^2$ – of C.





Let $D(x)$ be the real monic polynomial of degree $\sigma + \delta$, whose roots are the abscissae of the σ special crossings (in which the sign in the Conway sequence changes) and the abscissae of the δ nodes that are not crossings. A careful study of the sign alternations shows that

$$2b - 3 \leq \deg z(t)D(x(t)) = c + 3(\delta + \sigma) \leq c + 3(b - N - 1)$$

which is the announced result. $\square$

Sketch of Proof $b + c \geq 3N$

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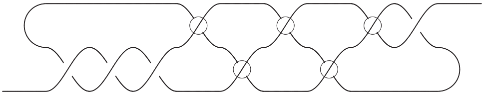

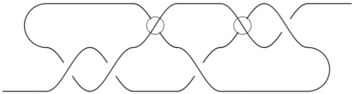

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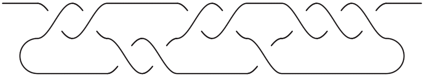
Consequence

Reduce to the study of trigonal plane curves of minimal degree b and the number of sign changes in the Gauss sequence.

Minimal diagrams vs of minimal degrees

$9_{15} = C(2, 2, 3, 2)$ of degree $\geq (3, 13, 14)$



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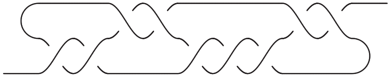

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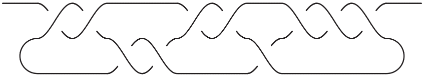

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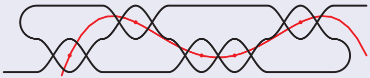
Bezout condition



At least 13 intersecting points $\Rightarrow \deg y \geq 13$.

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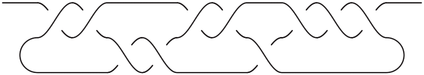

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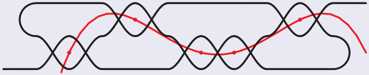

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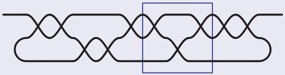
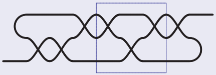
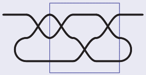
Adding a triple point (**T**-augmentation)



degree $(3,11)$



$\Leftarrow$ degree $(3,8)$



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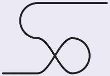
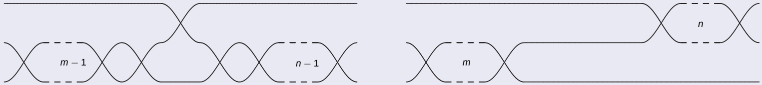


$\Leftarrow$ degree $(3,2)$

That proves that the lexicographic degree of 9_{15} is $(3,11,16)$.

Definition

Let x, y be (possibly empty) sequences of nonnegative integers and m, n be nonnegative integers. The plane diagram $D(x, m, n, y)$ is called a ***T**-reduction* of the diagram $D(x, m+1, 1, n+1, y)$.



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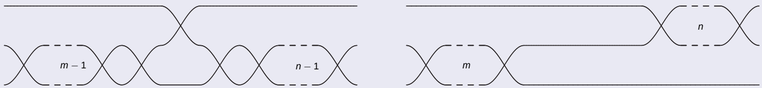


Proposition

*Let D_1 and D_2 be two plane trigonal diagrams such that D_2 is obtained from D_1 by a **T**-reduction. Suppose that there exists a trigonal polynomial curve of degree $(3, d-3)$ with diagram D_2. Then there exists a trigonal polynomial curve of degree $(3, d)$ that is $\mathcal{L}$-isotopic to D_1.*

Sketch of Proof:
Let us start with a polynomial curve $\mathcal{C} : x = P_3(t), y = Q_d(t)$ that is $\mathcal{L}$-isotopic to the plane diagram $D(u, m, n, v)$, where u, v are (possibly empty) sequences of nonnegative integers and m, n are nonnegative integers. By translation on x, we can suppose that $[x = 0]$ separates the m crossings from the n crossings. We can also suppose that $[x = 0]$ meets $\mathcal{C}$ in three points with nonzero y-coordinates. The curve (x, xy) will have the same double points as $\mathcal{C}$ and an additional triple point at $x = y = 0$. We claim that for ε small enough the curve $(P_3(t+\varepsilon), P_3(t) \cdot Q_d(t))$ will be $\mathcal{L}$-isotopic to either $D(u, m+1, 1, n+1, v)$ or $D(u, m, 1, 1, 1, n, v)$, depending on the sign of ε.

Example

We start with the polynomial parametrization $(T_3(t), T_4(t))$ of the trefoil $D(1,1,1)$. We choose to add a triple point in $(-3/4, 0)$ by considering the curve $x = T_3(t), y = Q_7(t)$ where $Q_7(t) = (T_3(t) + 3/4) \cdot (T_4(t) + 1)$.

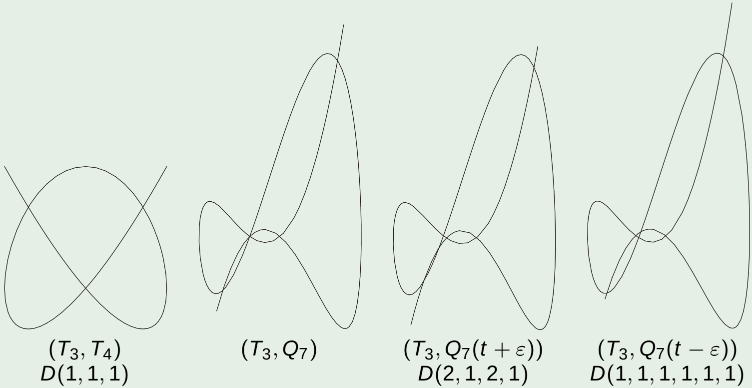
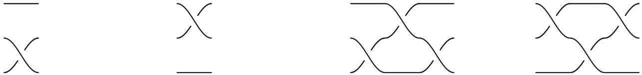


Figure: Adding three crossings to the trefoil

3-strands Braids



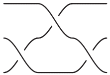
σ_1 $\quad$ σ_2 $\quad$ $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$

$$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$$

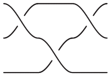
3-strands Braids

σ_1

σ_2



$\sigma_1\sigma_2\sigma_1$ $=$



$\sigma_2\sigma_1\sigma_2$

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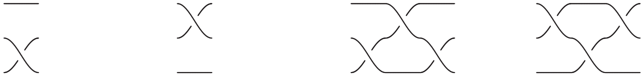
Associated braid

Let $\Phi : \mathbf{R}^4 \to \{(x, y) \in \mathbf{C}^2 \mid \mathrm{Im} x > 0\}$. Let $S_r = \partial B_r$ the image of the 3-sphere of radius r. If $C \subset \mathbf{C}^2$ is a real algebraic curve, then all links $S_r \cap C$ are isotopic if r is large enough.

The link $L_C = S_r \cap C$, for r large enough, is called the link associated to the real algebraic curve C.

3-strands Braids



σ_1 $\quad\quad$ σ_2 $\quad\quad$ $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$

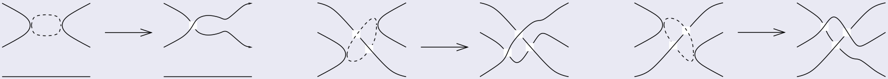
$$B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$$

Associated braid

Let $\Phi : \mathbf{R}^4 \to \{(x,y) \in \mathbf{C}^2 \mid \mathrm{Im} x > 0\}$. Let $S_r = \partial B_r$ the image of the 3-sphere of radius r. If $C \subset \mathbf{C}^2$ is a real algebraic curve, then all links $S_r \cap C$ are isotopic if r is large enough.

The link $L_C = S_r \cap C$, for r large enough, is called the link associated to the real algebraic curve C.

Algorithm (Orevkov)



a) $\supset_j \subset_j \to \sigma_j^{-1}$ $\quad$ b) $\supset_1 \subset_2 \to \sigma_1^{-1}\sigma_2^{-1}\sigma_1$ $\quad$ c) $\supset_2 \subset_1 \to \sigma_2^{-1}\sigma_1^{-1}\sigma_2$

Theorem (Orevkov)

If C is a rational real algebraic curve, then the associated braid must be a 3-component link, quasipositive with non negative linking numbers.

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Definition

A braid $\mathfrak{b} \in B_3$ is said to be *quasipositive* if it can be written in the form

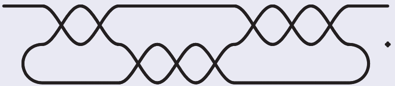
$$\mathfrak{b} = \prod_{i=1}^{l} w_i \sigma_1 w_i^{-1} \qquad \text{with } w_1, \cdots, w_l \in B_3. \tag{2}$$

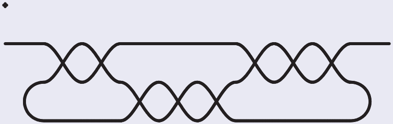
The quasipositivity problem in B_3 has been solved by Orevkov (2015).

The knot $8_6 = C(2,3,3)$

Looking for a degree 10 trigonal curve

A trigonal curve of degree $(3, b)$ has $b - 1$ double points. Such a curve has exactly one solitary node. Only two possibilities:





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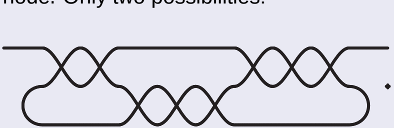

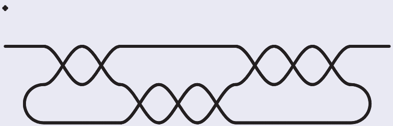

Associated braids

$b_1 = \sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-3}\sigma_2^{-3}\sigma_1^{-3}(\sigma_1\sigma_2\sigma_1)^4$

Linking numbers are $-1, 0, 1$

$b_2 = \sigma_2^{-2}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-3}\sigma_2^{-3}\sigma_1^{-2}(\sigma_1\sigma_2\sigma_1)^4$

Not a 3-components link

The knot $8_6 = C(2,3,3)$

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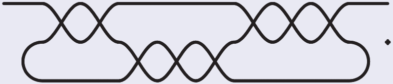

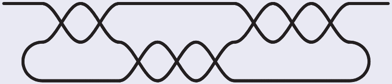

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$b_2 = \sigma_2^{-2}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-3}\sigma_2^{-3}\sigma_1^{-2}(\sigma_1\sigma_2\sigma_1)^4$
Not a 3-components link

The alternating diagram has degree $(3, 11, 13)$ at least.
$C(2, 2, 1, -4)$ is another diagram of 8_6. It can be reduced to the trefoil

$$D(2, \underline{2{,}1{,}4}) \longrightarrow D(\underline{2{,}1{,}3}) \longrightarrow D(1, 2)$$

by 2 **T**-reductions. It has degree $(3, 10, 14)$. 8_6 has degree $(3, 10, 14)$.

Strategy

For a given knot $K = C(m_1, \ldots, m_k)$,

- Compute an upper bound b_0 for $\deg y$ (Chebyshev diagram, see KP 2011).
    - Compute all diagrams with $b_0 - 1$ or less crossings. *Compute all continued fractions of length* $< b_0$.
- For each diagram,
    - Compute a lower bound b using Bézout-like boundaries.
    - Use possible **T**-reductions to get explicit constructions and upper bound.
    - If necessary, compute all possible braids associated to hypothetical plane curves of degree $b < b_0$ and test the conditions: quasipositivity, 3-components link, linking numbers.
- If the lower bound and the upper bound coincide, then we have determined the lexicographic degree $(3, b, c)$ of the knot.

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Example

$11_{a205} = C(2, 3, 1, 1, 1, 3)$.

$$D(2, 3, \underline{1,1,1}, 3) \longrightarrow D(2, 3, 3)$$

$\deg D(2, 3, 3) \geq (3, 11)$. Then we deduce $\deg D(2, 3, 1, 1, 1, 3) \geq (3, 14)$.
Another diagram is $C(2, 3, 1, 2, -4)$ and we have

$$D(2, \underline{3,1,2}, 4) \longrightarrow D(2, \underline{2,1,4}) \longrightarrow D(\underline{2,1,3}) \longrightarrow D(1, 2)$$

$\deg D(1, 2) = (3, 4)$. Then $\deg D(2, 3, 1, 2, 4) = (3, 13)$ and then 11_{a205} has degree $(3, 13, 20)$.

Preliminary diagrams

Fact (BKP 2016)

The degree of the torus knot $C(n)$ or $C(a,b)$ is $(3, \lfloor \frac{3N-1}{2} \rfloor, \lfloor \frac{3N}{2} \rfloor)$.

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The degree of the torus knot $C(n)$ or $C(a,b)$ is $(3, \lfloor \frac{3N-1}{2} \rfloor, \lfloor \frac{3N}{2} \rfloor)$.

- $b = 1$: $D(0,0)$
- $b = 2$: $D(1)$, $D(0,1)$
- $b = 4$: $D(0,2)$, $D(2,1)$
- $b = 5$: $D(0,1,1,0)$, $D(2,2)$, $D(1,1,1,1)$, $D(0,3)$, $D(1,2,0)$
- $b = 7$: $D(5)$, $D(1,4)$, $D(0,4)$

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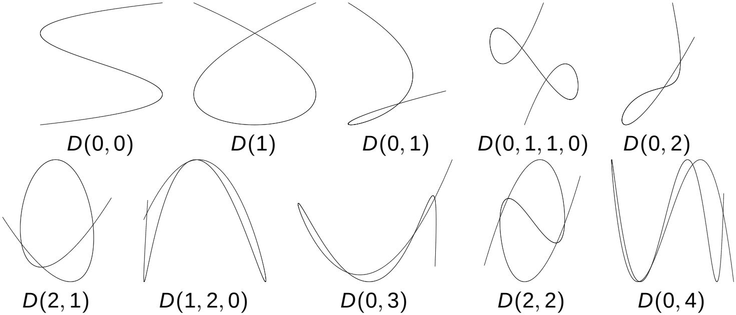

Results

Name	Fraction	Conway Not.	Lex. deg.	Cheb. deg.	diagram	Constr.
3_1	3	$C(3)$	$(3,4,5)$	4	$C(3)$	$D(3)$
4_1	$5/2$	$C(2,2)$	$(3,5,7)$	5	$C(2,2)$	$D(2,2)$
5_1	5	$C(5)$	$(3,7,8)$	7	$C(5)$	$D(5)$
5_2	$7/2$	$C(3,2)$	$(3,7,8)$	7	$C(3,1,1)$	$D(2,0)+\mathbf{T}$
6_1	$9/2$	$C(4,2)$	$(3,8,10)$	8	$C(4,2)$	$D(3,0)+\mathbf{T}$
6_2	$11/3$	$C(3,1,2)$	$(3,7,11)$	8	$C(3,1,2)$	$D(2,1)+\mathbf{T}$
6_3	$13/5$	$C(2,1,1,2)$	$(3,7,11)$	7	$C(2,1,1,2)$	$D(3)+\mathbf{T}$
7_1	7	$C(7)$	$(3,10,11)$	10	$C(7)$	$D(7)$
7_2	$11/2$	$C(5,2)$	$(3,10,11)$	10		Cheb.
7_3	$13/3$	$C(4,3)$	$(3,10,11)$	10		Cheb.
7_4	$15/4$	$C(3,1,3)$	$(3,8,13)$	10	$C(3,1,3)$	$D(2,2)+\mathbf{T}$
7_5	$17/5$	$C(3,2,2)$	$(3,10,11)$	10	$C(2,1,1,-4)$	$D(5)+\mathbf{T}$
7_6	$19/7$	$C(2,1,2,2)$	$(3,8,13)$	10		$D(0,1)+2\mathbf{T}$
7_7	$21/8$	$C(2,1,1,1,2)$	$(3,8,13)$	8		Cheb.
8_1	$13/2$	$C(6,2)$	$(3,11,13)$	11		Cheb.
8_2	$17/3$	$C(5,1,2)$	$(3,10,14)$	11		$D(4,1)+\mathbf{T}$
8_3	$17/4$	$C(4,4)$	$(3,11,13)$	11		Cheb.
8_4	$19/4$	$C(4,1,3)$	$(3,10,14)$	11	$C(4,1,2,1)$	$D(2,0)+2\mathbf{T}$
8_6	$23/7$	$C(3,3,2)$	$(3,10,14)$	11	$C(2,2,1,-4)$	$D(1,2)+2\mathbf{T}$
8_7	$23/5$	$C(4,1,1,2)$	$(3,10,14)$	10		Cheb.
8_8	$25/9$	$C(2,1,3,2)$	$(3,10,14)$	10		Cheb.
8_9	$25/7$	$C(3,1,1,3)$	$(3,10,14)$	11		$D(5)+\mathbf{T}$
8_{11}	$27/8$	$C(3,2,1,2)$	$(3,10,14)$	11		$D(2,0)+2\mathbf{T}$
8_{12}	$29/12$	$C(2,2,2,2)$	$(3,11,13)$	11		Cheb.
8_{13}	$29/8$	$C(3,1,1,1,2)$	$(3,10,14)$	10		Cheb.
8_{14}	$31/12$	$C(2,1,1,2,2)$	$(3,10,14)$	11		$D(2,0)+2\mathbf{T}$

Name	Fraction	Conway Not.	Lex. deg.	Cheb. deg.	diagram	Constr.
9_1	9	$C(9)$	$(3,13,14)$	13		Cheb.
9_2	$15/2$	$C(7,2)$	$(3,13,14)$	13		Cheb.
9_3	$19/3$	$C(6,3)$	$(3,13,14)$	13		Cheb.
9_4	$21/4$	$C(5,4)$	$(3,13,14)$	13		Cheb.
9_5	$23/4$	$C(5,1,3)$	$(3,11,16)$	13	$C(5,1,2,1)$	$D(3,0)+2\mathbf{T}$
9_6	$27/5$	$C(5,2,2)$	$(3,13,14)$	13		Cheb.
9_7	$29/9$	$C(3,4,2)$	$(3,13,14)$	13		Cheb.
9_8	$31/11$	$C(2,1,4,2)$	$(3,11,16)$	13	$C(2,1,4,1,1)$	$D(1,2,0)+2\mathbf{T}$
9_9	$31/7$	$C(4,2,3)$	$(3,13,14)$	13		Cheb.
9_{10}	$33/10$	$C(3,3,3)$	$(3,11,16)$	13	$C(3,2,1,-4)$	$D(0,1)+3\mathbf{T}$
9_{11}	$33/7$	$C(4,1,2,2)$	$(3,10,17)$	13		$D(3)+2\mathbf{T}$
9_{12}	$35/8$	$C(4,2,1,2)$	$(3,11,16)$	13		$D(3,0)+2\mathbf{T}$
9_{13}	$37/10$	$C(3,1,2,3)$	$(3,10,17)$	13		$D(1,2)+2\mathbf{T}$
9_{14}	$37/8$	$C(4,1,1,1,2)$	$(3,11,16)$	11		$D(3,0)+2\mathbf{T}$
9_{15}	$39/16$	$C(2,2,3,2)$	$(3,11,16)$	13	$C(2,2,2,1,-3)$	$D(1,0)+3\mathbf{T}$
9_{17}	$39/14$	$C(2,1,3,1,2)$	$(3,10,17)$	11		$D(3)+2\mathbf{T}$
9_{18}	$41/12$	$C(3,2,2,2)$	$(3,13,14)$	13		Cheb.
9_{19}	$41/16$	$C(2,1,1,3,2)$	$(3,11,16)$	11		$D(3,0)+2\mathbf{T}$
9_{20}	$41/11$	$C(3,1,2,1,2)$	$(3,10,17)$	13		$D(3)+2\mathbf{T}$
9_{21}	$43/12$	$C(3,1,1,2,2)$	$(3,11,16)$	13		$D(3,0)+2\mathbf{T}$
9_{23}	$45/19$	$C(2,2,1,2,2)$	$(3,10,17)$	13		$D(3)+\mathbf{T}$
9_{26}	$47/13$	$C(3,1,1,1,1,2)$	$(3,10,17)$	11		$D(3)+2\mathbf{T}$
9_{27}	$49/18$	$C(2,1,2,1,1,2)$	$(3,10,17)$	13		$D(3)+2\mathbf{T}$
9_{31}	$55/21$	$C(2,1,1,1,1,1,2)$	$(3,10,17)$	10		Cheb.

Results for the 186 2-bridge knots with 11 or fewer crossings (1873 diagrams to consider).

16 knots are such that the alternating diagram is not of lexicographic degree.

References

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- E. Brugallé, P. -V. Koseleff, D. Pecker, *On the lexicographic degree of two-bridge knots*, Ann. Fac. Sciences Toulouse, to appear, 30p.

Erwan BRUGALLÉ
Université de Nantes, Centre Mathématiques Jean Leray
e-mail: `erwan.brugalle@math.cnrs.fr`

Pierre-Vincent KOSELEFF
Université Pierre et Marie Curie (UPMC Sorbonne Université),
Institut de Mathématiques de Jussieu (IMJ-PRG) & Inria-Paris
e-mail: `pierre-vincent.koseleff@upmc.fr`

Daniel PECKER
Université Pierre et Marie Curie (UPMC Sorbonne Université),
Institut de Mathématiques de Jussieu (IMJ-PRG),
e-mail: `danielpecker@wanadoo.fr`