Structured Low Rank Decomposition, Completion and Applications

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Applications and Motivations

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  - Signal Processing,
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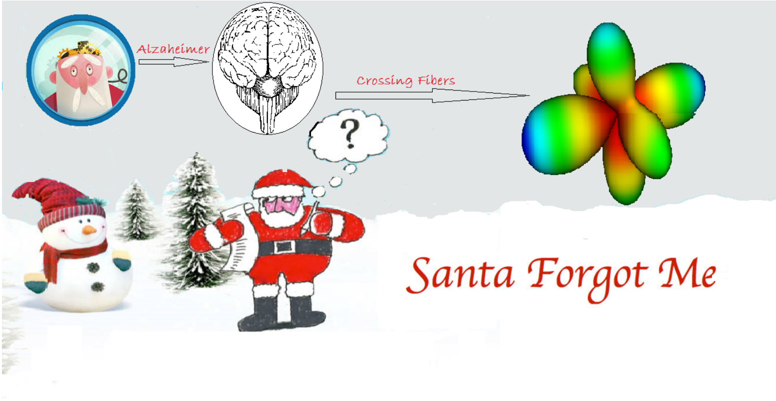
  - The Analysis of Contents of Web Pages Model.

Fiber Crossing Detection Model



Modeling the Fibers Orientation Function **ODF** which describes the diffusion of white matter in brain, by a tensor T of dimension 3 and order 4.

ODF Tensor:Numerical Results

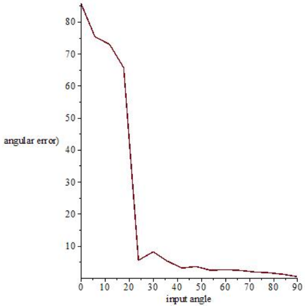


Figure: Angular error between input and output directions

The Analysis of contents of Web Pages Model

- Collection: N documents
- What is the topic of a document?
- Each document is represented by a vector $c \in \mathbb{R}^n$ where each component is the occurence of a word drawn from a vocabulary which contains n words. This is a sparse vector.

Learning Phase

Theorem: Let x_1, x_2, x_3 be the first three words of a corpus independently drawn from a discrete distribution specified by the $\sum_{i=1}^{r} h_i \xi_i$. If

$$M_1 := P(x_1) \tag{1}$$

$$M_2 := P(x_1 \otimes x_2) - \frac{\alpha_0}{\alpha_0 + 1} M_1 \otimes M_1 \tag{2}$$

and

$$\begin{aligned} M_3 = {} & P(x_1 \otimes x_2 \otimes x_3) \\ & - \frac{\alpha_0}{\alpha_0 + 2} \left(P(M_1 \otimes x_1 \otimes x_2) + P(x_1 \otimes M_1 \otimes x_2) + P(x_1 \otimes x_2 \otimes M_1) \right) \\ & + \frac{2\alpha_0^2}{(\alpha_0 + 2)(\alpha_0 + 1)} M_1 \otimes M_1 \otimes M_1 \end{aligned} \tag{3}$$

then

$$M_2 := \sum_{p=1}^{r} \frac{\alpha_p}{(\alpha_0 + 1)(\alpha_0)} (\xi_p)^2$$

and

$$M_3 := \sum_{p=1}^{r} \frac{2\alpha_p}{(\alpha_0 + 2)(\alpha_0 + 1)\alpha_0} (\xi_p)^3$$

where $\xi_p = (\xi_{p,1}, \ldots, \xi_{p,n}) \in \mathbb{R}^n$.

Learning Phase

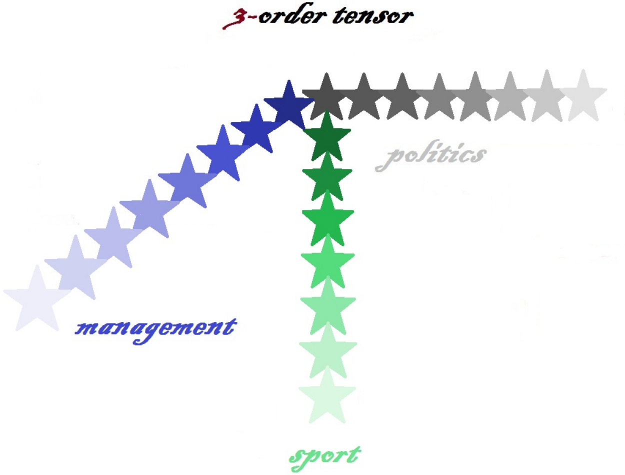
* We compute the 1-order tensors associated to all c vectors M_{1c} and we compute the mean $M_1 \in \mathbb{R}^n$ of all of them.

* We compute the 2-order tensors associated to all c vectors M_{2c} and we compute the mean $M_2 \in \mathbb{R}^{n*n}$ of all of them.

* We compute the singular value decomposition of $M_2 = USU^T$ and the whitening matrix $W = U_r S_r^{-\frac{1}{2}} \in \mathbb{R}^{n*r}$

* We compute the 3-order compressed tensors $K_{3c} \in \mathbb{R}^{r*r*r}$:
  $K_{3c} = (W^T, W^T, W^T).M_{3c}$ where $M_{3c} \in \mathbb{R}^{n*n*n}$ and we compute the mean $K_3 \in \mathbb{R}^{r*r*r}$ of all of them. .

Learning Phase

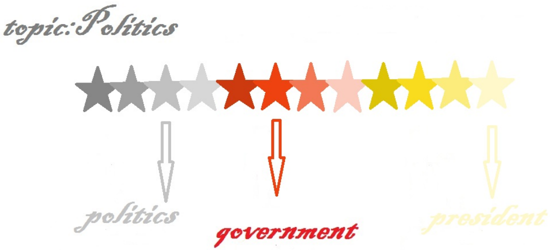


* We decompose the tensor $K3 = \sum w_i u_i \otimes u_i \otimes u_i$ and deduce the decomposition $M3 = \sum w_i \mu_i \otimes \mu_i \otimes \mu_i$ where $\mu_i = W u_i$

Learning Phase

∗ Each topic is represented by a probability vector and each component of it is equal to the probability of a word belongs to this topic

∗ Project a vector c corresponding to a fixed document on the new basis of vectors to compute the weights of each topic in the document.




10000 Documents
100 Words
r = 3
a0 = 0.01
K3[:,:,1]
1.0153
-0.0094
-0.0071
-0.0094
1.0078
-0.0054
-0.0071
-0.0054
1.0209
K3[:,:,2]
-0.0094
1.0078
-0.0054
1.0078
0.1942
0.6905
-0.0054
0.6905
-0.1454
K3[:,:,3]
-0.0071
-0.0054
1.0209
-0.0054
0.6905
-0.1454
1.0209
-0.1454
-0.7283

Points =

$\mathbf{u}_i$	$a_{i,0}$	$a_{i,1}$	$a_{i,2}$	$b_{i,0}$	$b_{i,1}$	$b_{i,2}$	$c_{i,0}$	$c_{i,1}$	$c_{i,2}$
$\mathbf{u}_1$	1.0	-0.0936	-1.4190	1.0	-0.0936	-1.4199	1.0	-0.0936	-1.4190
$\mathbf{u}_2$	1.0	-1.1394	0.7793	1.0	-1.1394	0.7790	1.0	-1.1394	0.7793
$\mathbf{u}_3$	1.0	1.3	0.6183	1.0	1.3	0.6182	1.0	1.3	0.6183

Weights =

ω_1	0.3394
ω_2	0.3510
ω_3	0.3248

$\mathbf{r} = 2$

Dual space

Power Formal series :

$$\sigma(\mathbf{y}) = \sum_{\alpha \in \mathbb{N}^n} \sigma_\alpha \frac{\mathbf{y}^\alpha}{\alpha!} \in \mathbb{C}[[\mathbf{y}]]$$

Linear Functional :

$$\begin{array}{rcl} \sigma : \mathbb{C}[\mathbf{x}] & \rightarrow & \mathbb{C} \\ p = \sum_{\alpha \in A \subset \mathbb{N}^n} p_\alpha \mathbf{x}^\alpha & \mapsto & \langle \sigma \mid p \rangle = \sum_{\alpha \in A \subset \mathbb{N}^n} p_\alpha \sigma_\alpha . \end{array}$$

Duality :

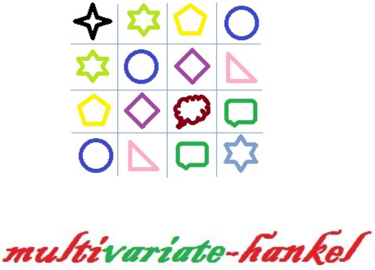
$$\mathbb{C}^{\mathbb{N}^n} \equiv \mathbb{C}[\mathbf{x}]^* \equiv \mathbb{C}[[\mathbf{y}]]$$

$$L_0(\mathbb{C}^{\mathbb{N}^n}) \equiv \mathbb{C}[\mathbf{x}]$$

$\rightsquigarrow$ *Hankel Operator* :

$$\begin{array}{rcl} H_\sigma : \mathbb{C}[\mathbf{x}] & \rightarrow & \mathbb{C}[[\mathbf{y}]] \\ p & \mapsto & p \star \sigma \end{array}$$

Hankel Matrices



Hankel Matrices:

$$H = [\sigma_{i+j}]_{0\le i\le l, 0\le j\le m}$$

Multivariate Hankel Matrices:

$$H = [\sigma_{\alpha+\beta}]_{\alpha\in A,\beta\in B}$$

Multivariate Hankel operators: $\sigma = (\sigma_\alpha)_{\alpha\in\mathbb{N}^n} \in \mathbb{C}^{\mathbb{N}^n} \rightsquigarrow$

$$\begin{aligned} H_\sigma : L_0(\mathbb{C}^{\mathbb{N}^n}) &\to \mathbb{C}^{\mathbb{N}^n} \\ (p_\alpha)_\alpha &\mapsto \Big(\sum_\alpha p_\alpha \sigma_{\alpha+\beta}\Big)_{\beta\in\mathbb{N}^n} \end{aligned} \qquad (4)$$

The moment matrix $H_{\sigma}^{A,B}$ of σ: $(\mathbf{x}^{\beta})_{\beta\in B}$ and $(\frac{\mathbf{y}^{\alpha}}{\alpha!})_{\alpha\in A} \rightsquigarrow$

$$H_{\sigma}^{A,B} = [\sigma_{\alpha+\beta}]_{\alpha\in A,\beta\in B}.$$

The evaluation at ξ:

$$\mathbf{e}_{\xi}(\mathbf{y}) = \sum_{\alpha\in\mathbb{N}^n} \xi^{\alpha} \frac{\mathbf{y}^{\alpha}}{\alpha!} = e^{\mathbf{y}\cdot\xi}$$
$$\rightsquigarrow \forall p \in R, < \mathbf{e}_{\xi} | p >= \sum_{\alpha\in\mathbb{N}^n} p_{\alpha}\, \xi^{\alpha} = p(\xi).$$

- $H_{\mathbf{e}_{\xi}} : p \mapsto p \star \mathbf{e}_{\xi} = p(\xi)\mathbf{e}_{\xi}$,
- ${H_{\xi}}^{A,B} = [\xi^{\beta+\alpha}]_{\beta\in B,\alpha\in A}$, if ${H_{\xi}}^{A,B} \neq 0$

Decomposition of Hankel matrices

Decomposition of Quotient Algebra

Theorem:When the roots are simple, I_σ kernel of H_σ :: $\mathcal{A}_\sigma = \mathbb{C}[\mathbf{x}]/I_\sigma$ quotient algebra $\rightsquigarrow$

- $\sigma = \sum_{i=1}^{r} \omega_i \mathbf{e}_{\xi_i}(\mathbf{y})$ with $\omega_i \in \mathbb{C}$ are non-zeros, $\xi_i \in \mathbb{C}^n$ where ξ_i are distinct.
- H_σ is of rank r and the roots of multiplicity one.
- $(\mathbf{e}_{\xi_1}, \dots, \mathbf{e}_{\xi_r})$ is a basis of $\mathcal{A}_\sigma^\star$.

The decomposition problem σ as a weighted sum of products of power of linear forms reduces to the solution of the polynomial equations $p = 0$ for p in the kernel I_σ of H_σ.

$\mathbb{C}[\mathbf{x}]/I_\sigma$ is Artinian :: finite dimension over $\mathbb{C}$

$\rightsquigarrow$ I_σ and $\mathcal{V}(I_\sigma) = \{\xi_1, \dots, \xi_r\} = \{\xi \in \mathbb{C}^n \mid \forall p \in I_\sigma, p(\xi) = 0\}$
$\rightsquigarrow$ decomposition of $\mathcal{A}$ as a sum of sub-algebras:

$$\mathcal{A} = \mathbb{C}[\mathbf{x}]/I_\sigma = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_r$$

where $\mathcal{A}_i = \mathbf{u}_{\xi_i}\mathcal{A} \sim \mathbb{C}[\mathbf{x}]/Q_i$ and Q_i is the primary component of I associated to the root $\xi_i \in \mathbb{C}^n$.

The idempotents $\mathbf{u}_{\xi_1}, \dots, \mathbf{u}_{\xi_r}$ :: $\mathbf{u}_{\xi_i}^2(\mathbf{x}) \equiv \mathbf{u}_{\xi_i}(\mathbf{x})$, $\sum_{i=1}^{r} \mathbf{u}_{\xi_i}(\mathbf{x}) \equiv 1$.

Decomposition of Hankel matrices

Multiplication Operator

Multiplication Operator: $g \in \mathbb{C}[\mathbf{x}]$, $\mathcal{M}_g$

$$\begin{array}{rccl} \mathcal{M}_g: & \mathcal{A} & \to & \mathcal{A} \\ & h & \mapsto & \mathcal{M}_g(h) = g\,h. \end{array}$$

Transpose of Multiplication Operator:

$$\begin{array}{rccl} \mathcal{M}_g{}^{\mathsf{T}}: & \mathcal{A}^* & \to & \mathcal{A}^* \\ & \Lambda & \mapsto & \mathcal{M}_g{}^{\mathsf{T}}(\Lambda) = \Lambda \circ \mathcal{M}_g = g \star \Lambda. \end{array}$$

Decomposition of Hankel matrices

Eigenvalues and eigenvectors

Theorem: Let I ideal of $\mathbb{C}[\mathbf{x}]$ and $\mathcal{V}(I) = \{\xi_1, \xi_2, \ldots, \xi_r\}$ such that ξ_i are simple $\Rightarrow$

- $\forall g \in \mathcal{A}$, the eigenvalues of $\mathcal{M}_g$ and $\mathcal{M}_g{}^\intercal$ are the values $g(\xi_1), \ldots, g(\xi_r)$ of the polynomial g at the roots with multiplicities $\mu_i = \dim \mathcal{A}_i = 1$.
- The eigenvectors common to all $\mathcal{M}_g{}^\intercal$ with $g \in \mathcal{A}$ are - up to a scalar - the evaluations $\mathbf{e}_{\xi_1}, \ldots, \mathbf{e}_{\xi_r}$.
- If g is separating the roots, i.e. $g(\xi_p) \neq g(\xi_q)$ for $p \neq q$, then the eigenvectors of $\mathcal{M}_g$ are, up to a scalar, interpolation polynomials $\mathbf{u}_{\xi_i}$ at the roots ξ_i.

Decomposition of Hankel matrices

Bases

Lemma:Let $B = \{b_1, \ldots, b_r\}$, $B' = \{b'_1, \ldots, b_r{}'\} \subset \mathbb{C}[\mathbf{x}]$. If the matrix $H_\sigma^{B,B'} = (\langle \sigma | b_i b'_j \rangle)_{1 \leq i,j \leq r}$ is invertible $\Rightarrow$ B and B' are linearly independent in $\mathcal{A}_\sigma$.

Decomposition of Hankel matrices

Multiplication operator via truncated Hankel Matrices

Proposition: Let B, B' be basis of $\mathcal{A}_\sigma$ and $g \in \mathbb{C}[\mathbf{x}]$. We have

$$H_{g\star\sigma}^{B,B'} = ({M_g}^{B})^{\mathsf{T}} H_{\sigma}^{B,B'} = H_{\sigma}^{B,B'} {M_g}^{B'}. \tag{5}$$

where ${M_g}^{B}$ (resp. ${M_g}^{B'}$) is the matrix of the multiplication by g in the basis B (resp. B') of $\mathcal{A}_\sigma$.

Let $\sigma(\mathbf{y}) = \sum_{i=1}^{r} \omega_i \mathbf{e}_{\xi_i}(\mathbf{y})$ with $\omega_i \in \mathbb{C} \setminus \{0\}$ and $\xi_i \in \mathbb{C}^n$ distinct and simple.
Let B, B' be bases of $\mathcal{A}_\sigma$ $\rightsquigarrow$

- For $g \in \mathbb{C}[\mathbf{x}]$, ${M_g}^{B'} = (H_{\sigma}^{B,B'})^{-1} H_{g\star\sigma}^{B,B'}$, $({M_g}^{B})^{\mathsf{T}} = H_{g\star\sigma}^{B,B'} (H_{\sigma}^{B,B'})^{-1}$.
- For $g \in \mathbb{C}[\mathbf{x}]$, the generalized eigenvalues of $(H_{g\star\sigma}^{B,B'}, H_{\sigma}^{B,B'})$ are $g(\xi_i)$ with multiplicity 1, $i = 1, \ldots, r$.
- The generalized eigenvectors common to all $(H_{g\star\sigma}^{B,B'}, H_{\sigma}^{B,B'})$ for $g \in \mathbb{C}[\mathbf{x}]$ are - up to a scalar - $(H_{\sigma}^{B,B'})^{-1} B(\xi_i)$, $i = 1, \ldots, r$.

Singular Value Decomposition

⋆ $\sigma_\alpha, |\alpha| \leq d, d_1 + d_2 \leq d$, $A_1 = (\mathbf{x}^\alpha)_{|\alpha| \leq d_1}$ and $A_2 = (\mathbf{x}^\beta)_{|\beta| \leq d_2}$

Truncated Hankel operator associated to σ is:

$$\begin{array}{rcccc} H_\sigma^{d_1,d_2} & : & \mathbb{C}[\mathbf{x}]_{d_2} & \rightarrow & (\mathbb{C}[\mathbf{x}]_{d_1})^* \\ & & p & \mapsto & p \star \sigma \end{array}$$

Truncated Hankel matrix $H_\sigma^{d_1,d_2}$
Singular Value Decomposition

$$H_\sigma^{d_1,d_2} = USV^\mathsf{T}$$

⋆ Vectors: $u_i = [u_{\alpha,i}]_{\alpha \in A_1}$, $v_j = [v_{\beta,j}]_{\beta \in A_2}$ :: i^{th} and j^{th} col of U^H and $\overline{V}$
⋆ Polynomials: $u_i(\mathbf{x}) = u_i^\mathsf{T} A_1 = \sum_{|\alpha| \leq d_1} u_{\alpha,i} \mathbf{x}^\alpha$ and $v_j(\mathbf{x}) = v_j^\mathsf{T} A_2 = \sum_{|\beta| \leq d_2} v_{\beta,j} \mathbf{x}^\beta$.
⋆ Bases: $U_r{}^\mathsf{H} := (u_i(\mathbf{x}))_{i=1,\ldots,r}$ and $\overline{V}_r := (v_j(\mathbf{x}))_{j=1,\ldots,r}$

Multiplication in the orthogonal basis

and computation of weights

Proposition: If rank $H_{\sigma}^{d_1,d_2} = r$,

$\star$the sets of polynomials $U_r{}^{\mathsf{H}}$ and $\overline{V}_r$ are bases of $\mathcal{A}_\sigma$.
$\star$The matrix $M_{x_i}^{\overline{V}_r}$ associated to the multiplication operator by x_i in the basis $\overline{V}_r$ of $\mathcal{A}_\sigma$ is $M_{x_i}^{\overline{V}_r} = S_r{}^{-1}\, U_r{}^{\mathsf{H}}\, H_{x_i\star\sigma}^{d_1,d_2}\, \overline{V}_r$ $i = 1, \ldots, n$.

Proposition: Let $\sigma = \sum_{i=1}^{r} \omega_i\, \mathbf{e}_{\xi_i}$ with $\omega_i \in \mathbb{C}\setminus\{0\}$, $\xi_i = (\xi_{i,1}, \ldots, \xi_{i,n}) \in \mathbb{C}^n$ and $M_{x_j}^{\overline{V}_r}$ be the matrix of multiplication by x_j in the basis $\overline{V}_r$. Let $\mathbf{v}_i$ be a common eigenvector of $M_{x_j}^{\overline{V}_r}$, $j = 1, .., n$ for the eigenvalues $\xi_{i,j}$. $\Rightarrow$ the weight of $\mathbf{e}_{\xi_i}$ in the decomposition of σ is

$$\omega_i = \frac{[1]^{\mathsf{T}}\, H_{\sigma}^{d_1,d_2}\, \overline{V}_r\, \mathbf{v}_i}{[\xi_i^{\alpha}]_{\alpha\in A_2}^{\mathsf{T}}\, \overline{V}_r\, \mathbf{v}_i}.$$

Algorithm 3.1: Decomposition of polynomial-exponential series with constant weights

Input: the moments σ_α of σ for $|\alpha| \leq d$.
Let d_1 and d_2 be positive integers such that $d_1 + d_2 + 1 = d$, for example $d_1 := \lceil \frac{d-1}{2} \rceil$ and $d_2 := \lfloor \frac{d-1}{2} \rfloor$.

1. Compute the Hankel matrix $H_\sigma^{d_1,d_2} = [\sigma_{(\alpha+\beta)}]_{\substack{|\alpha|\leq d_1\\ |\beta|\leq d_2}}$ of σ in for the monomial sets $A_1 = (\mathbf{x}^\alpha)_{|\alpha|\leq d_1}$ and $A_2 = (\mathbf{x}^\beta)_{|\beta|\leq d_2}$.
2. Compute the singular value decomposition of $H_\sigma^{d_1,d_2} = USV^{\mathbf{T}}$ with singular values $s_1 \geq s_2 \geq \cdots \geq s_m \geq 0$.
3. Determine its numerical rank, that is, the largest integer r such that $\frac{s_r}{s_1} \geq \epsilon$.
4. Form the matrices $M_{x_i}^{\overline{V}_r} = {S_r}^{-1} {U_r}^{\mathbf{H}} H_{x_i\star\sigma}^{d_1,d_2} \overline{V}_r$, $i = 1, \ldots, n$, where $H_{x_i\star\sigma}^{d_1,d_2}$ is the Hankel matrix associated to $x_i \star \sigma$.
5. Compute the eigenvectors $\mathbf{v}_j$ of $\sum_i^n l_i M_{x_i}$ for a random choice of l_i in $[-1, 1]$, $i = 1, \ldots, n$ and for each $j = 1, \ldots, r$ do the following:
   - *a.* Compute $\xi_{j,i}$ such that $M_i \mathbf{v}_j = \xi_{j,i} \mathbf{v}_j$ for $i = 1, \ldots, n$ and deduce the point $\xi_j := (\xi_{j,1}, \ldots, \xi_{j,n})$.
   - *b.* Compute $\omega_j = \frac{<\sigma|\mathbf{v}_j(\mathbf{x})>}{\mathbf{v}_j(\xi_j)} = \frac{[1]^{\mathbf{T}}\, H_\sigma^{d_1,d_2}\, \overline{V}_r\, \mathbf{v}_j}{[\xi_i^\alpha]^{\mathbf{T}}_{\alpha\in A_2} \overline{V}_r\, \mathbf{v}_j}$ where $[1]$ is the coefficient vector of 1 in the basis A_1.

Output: $r \in \mathbb{N}$, $\omega_j \in \mathbb{C}\backslash(0)$, $\xi_j \in \mathbb{C}^n$, j=1, . . . , r such that $\sigma(\mathbf{y}) = \sum_{j=1}^r \omega_j\, \mathbf{e}_{\xi_j}(\mathbf{y})$ up to degree d.

Decomposition Algorithm:Numerical Results

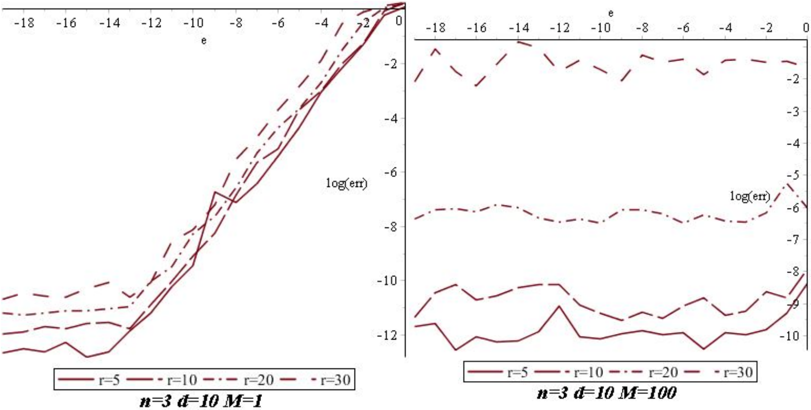


Figure: The evolution of the error in terms of the perturbation $\epsilon = 10^{(-e)}$ on the moments of the form $\sigma_\alpha + \epsilon(p_\alpha + iq_\alpha)$ with amplitude M for different values of r.

Decomposition Algorithm:Numerical Results

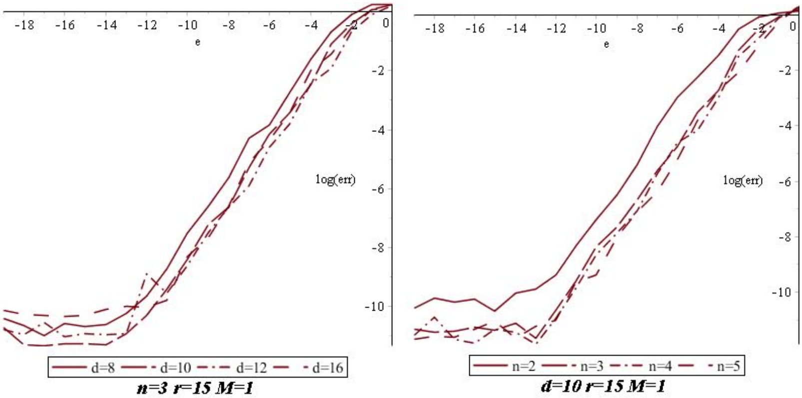


Figure: The evolution of the error in terms of the perturbation $\epsilon = 10^{(-e)}$ on the moments of the form $\sigma_\alpha + \epsilon(p_\alpha + iq_\alpha)$ with amplitude $M = 1$ for different values of d and n.

Decomposition Algorithm:Numerical Results

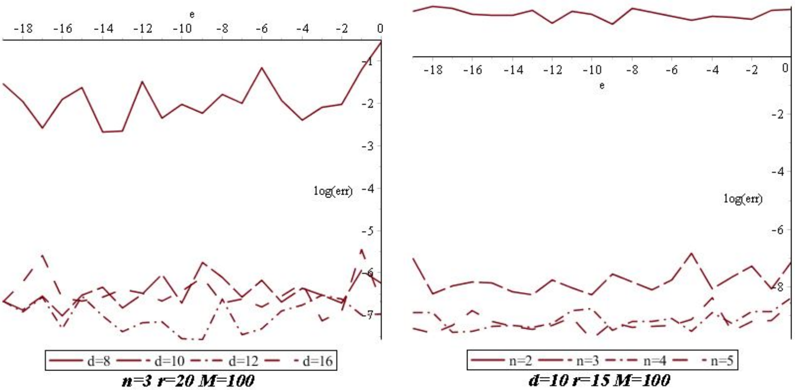


Figure: The evolution of the error in terms of the perturbation $\epsilon = 10^{(-e)}$ on the moments of the form $\sigma_\alpha + \epsilon(p_\alpha + iq_\alpha)$ with amplitude $M = 10$ for different values of d and n.

Rescaling

⋆ $\lambda \neq 0$,

$$\sigma(\mathbf{y}) := \sum_{\alpha \in N^n} \sigma_\alpha \frac{\mathbf{y}^\alpha}{\alpha!} \longrightarrow \tilde{\sigma}(\mathbf{y}) := \sigma(\lambda \mathbf{y}) = \sum_{\alpha \in N^n} \lambda^{|\alpha|} \sigma_\alpha \frac{\mathbf{y}^\alpha}{\alpha!},$$

$\rightsquigarrow$ $e_\xi(\lambda \mathbf{y}) = e_{\lambda\xi}(\mathbf{y})$.

⋆ Decomposition of $\tilde{\sigma}(\mathbf{y}) = \sigma(\lambda \mathbf{y})$ from the moments $\tilde{\sigma}_\alpha = \lambda^{|\alpha|}\sigma_\alpha$.

⋆ Inverse Scaling of frequencies $\tilde{\xi}_i :: \xi_i = \frac{\tilde{\xi}_i}{\lambda} = \left(\frac{\tilde{\xi}_{i,1}}{\lambda}, \ldots, \frac{\tilde{\xi}_{i,n}}{\lambda}\right)$.

$*\lambda := \frac{1}{m}$ where $m = \frac{max_{|\alpha|=d}|\sigma_\alpha|}{max_{|\alpha|=d-1}|\sigma_\alpha|}$

Rescaling numerical influence

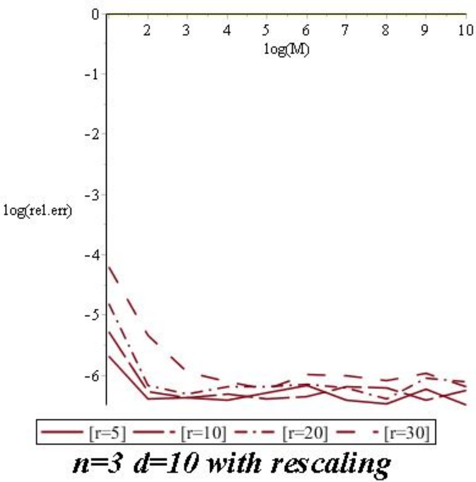


Figure: The evolution of the relative error in terms of the amplitude M for different values of r.

Multi-symmetric tensor

$(E_j)_{1\leq j\leq k}|dim(E_j) = n_j + 1, E_j = \langle \mathbf{x}_j \rangle = \langle x_j, \ldots, x_{j,n_j} \rangle$.

$\mathcal{S}^{\delta_j}(E_j) = \{p(\mathbf{x}_j)\text{homogeneous}, degree(p(\mathbf{x}_j)) = \delta_j\}$.

$\mathcal{S}^{\delta}(E) = \mathcal{S}^{\delta_1}(E_1) \otimes \mathcal{S}^{\delta_2}(E_2) \otimes \ldots \otimes \mathcal{S}^{\delta_k}(E_k)$.
$[T] \in \mathcal{S}^{\delta}(E)$ is a multi symmetric tensor.

Notation: $\underline{\mathbf{x}} = (\underline{x}_1, \ldots, \underline{x}_k)$ and $\alpha = (\alpha_1, \ldots, \alpha_k)$ and
$\underline{x}_j = (x_{j,1}, \ldots, x_{j,n_j})$ and $\alpha_j = (\alpha_{j,0}, \ldots, \alpha_{j,n_j})$
so that $\underline{T}(\underline{\mathbf{x}}_1, \ldots, \underline{\mathbf{x}}_k) = \sum_{\alpha_j \in \mathbb{N}^{n_j+1}, |\alpha_j| \leq \delta_j} t_{\alpha_1,\ldots,\alpha_k} \underline{\mathbf{x}}^{\alpha}$ where
$\underline{\mathbf{x}}^{\alpha} = \prod_{j=1}^{k} \prod_{p=1}^{n_j} x_{j,p}{}^{\alpha_{j,p}}$

Multi-symmetric tensor

$T(\mathbf{x}) = T(\mathbf{x}_1, \ldots, \mathbf{x}_k)$: an multi-homogeneous polynomial of degree δ_j in the variable $\mathbf{x}_j = (x_{j,0}, \ldots, x_{j,n_j}) \rightsquigarrow$
$[T] = [t_{\alpha'_1, \alpha'_2, \ldots, \alpha'_k}]_{\substack{|\alpha'_j| = \delta_j \\ \alpha'_j \in \mathbb{N}^{n_j+1}}}$: multi symmetric array of coefficients such that each $\alpha'_j = (\alpha'_{j,p_j})_{0 \leq p_j \leq n_j}$ is a multi-index for $1 \leq j \leq k$.

Let $u_{i,j,0} \neq 0, j = 1, \ldots, k, i = 1, \ldots, r$,
then for $(u_{i,j,0})' = 1$ and $x_{j,0} = 1 \rightsquigarrow$

- $R = \mathbb{C}[\underline{\mathbf{x}}_1, \underline{\mathbf{x}}_2, \ldots, \underline{\mathbf{x}}_k]$ where $\underline{\mathbf{x}}_j = (x_{j,1}, \ldots, x_{j,n_j})$ for $j = 1, \ldots, k$
- $R_{\delta_1, \delta_2, \ldots, \delta_k} = \{T \in \mathcal{S}^{\delta}(E) | x_{j,0} = 1, j = 1, \ldots, k\}$

Multi Symmetric Tensor Decomposition

Sum of products of power of linear forms:

$T(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k) = \sum_{p=1}^{r} \omega_p \mathbf{u}_{p,1}^{\delta_1}(\mathbf{x}_1)\mathbf{u}_{p,2}^{\delta_2}(\mathbf{x}_2)\ldots\mathbf{u}_{i,k}^{\delta_k}(\mathbf{x}_k)$ where
$\mathbf{u}_{p,j}(\mathbf{x}_j) = u_{p,j}x_j + u_{p,j,1}x_{j,1} + \ldots + u_{p,j,n_j}x_{j,n_j}$ and
$\mathbf{u}_p = (u_{p,j,p_j})_{\substack{0 \le p_j \le n_j \\ 1 \le j \le k}} \in \mathbb{C}^{\sum_{j=1}^{k}(n_j+1)}$ is the coefficient vector associated to the linear forms $\mathbf{u}_{p,j}(\mathbf{x}_j)$ in the basis $\mathbf{x}_j$ for $j = 1, \ldots, k$.

Rank r of T: Minimal number of terms in a decomposition of $T(\mathbf{x})$.

For $|\alpha_j| \le \delta_j$, we denote $\bar{\alpha}_j := (\delta_j - |\alpha_j|, \alpha_{j,1}, \ldots, \alpha_{j,n_j}), j = 1, \ldots, k$
We identify $t_{\alpha_1', \ldots, \alpha_k'} := t_{\bar{\alpha}_1, \ldots, \bar{\alpha}_k}$.

By a generic change of coordinates in each E_j, we may assume that $u_{p,j} \neq 0$ and that T has an affine decomposition. Then by scaling $\mathbf{u}_p(\mathbf{x})$ and multiplying ω_p by the δ^{th} power of the scaling factor we may assume that $u_{p,j} = 1$ for $p = 1, \ldots, r$ and $j = 1, \ldots, k$. Thus the polynomial
$\underline{T}(\underline{\mathbf{x}}) = \sum_{p=1}^{r} \omega_p' \mathbf{u}_p'^{\delta}(\underline{\mathbf{x}}) = \sum_{p=1}^{r} \omega_i' {\mathbf{u}_{p,1}'}^{\delta_1}(\underline{\mathbf{x}}_1){\mathbf{u}_{p,2}'}^{\delta_2}(\underline{\mathbf{x}}_2)\ldots{\mathbf{u}_{p,k}'}^{\delta_k}(\underline{\mathbf{x}}_k)$

$T_1(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k),\ T_2(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k) \in \mathcal{S}^\delta(E) \rightsquigarrow$
Apolar Product:

$$\langle \underline{T_1}(\underline{\mathbf{x}}_1, \underline{\mathbf{x}}_2, \ldots, \underline{\mathbf{x}}_k), \underline{T_2}(\underline{\mathbf{x}}_1, \underline{\mathbf{x}}_2, \ldots, \underline{\mathbf{x}}_k)\rangle = \sum_{\substack{|\alpha_j| \le \delta_j \\ \alpha_j \in \mathbb{N}^{n_j}}} \sigma^{(1)}_{\alpha_1, \alpha_2, \ldots, \alpha_k} \bar{\sigma}^{(2)}_{\alpha_1, \alpha_2, \ldots, \alpha_k} \binom{\delta}{\alpha}$$

where $\binom{\delta}{\alpha} = \binom{\delta_1}{\alpha_1}\binom{\delta_2}{\alpha_2} \ldots \binom{\delta_k}{\alpha^k}$.

Dual Operator:

$$T^* : (R_{\delta_1, \delta_2, \ldots, \delta_k}) \to (R_{\delta_1, \delta_2, \ldots, \delta_k})^* \quad (6)$$

$$\underline{T_2} \mapsto T^*(\underline{T_2}) = \langle \underline{T}(\underline{\mathbf{x}}), \underline{T_2}(\underline{\mathbf{x}})\rangle \quad (7)$$

For $T = (t_{\alpha_1, \alpha_2, \ldots, \alpha_k})_{\substack{|\alpha_j| \le \delta_j \\ \alpha_j \in \mathbb{N}^{n_j}}} \in \mathcal{S}^\delta(E) \rightsquigarrow$

- $\sigma_{\alpha_1, \alpha_2, \ldots, \alpha_k}(T) = \sigma_{\alpha_1, \alpha_2, \ldots, \alpha_k} = t_{\alpha_1, \alpha_2, \ldots, \alpha_k} \binom{\delta_1}{\alpha_1}^{-1} \binom{\delta_2}{\alpha_2}^{-1} \ldots \binom{\delta_k}{\alpha_k}^{-1}$.
- Dual via the formal power series:

$$T^*(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k) = \sum_{\substack{|\alpha_j| \le \delta_j \\ \alpha_j \in \mathbb{N}^{n_j}}} \sigma_{\alpha_1, \alpha_2, \ldots, \alpha_k} \frac{(\mathbf{y}_1)^{\bar{\alpha}_1}}{\bar{\alpha}_1!} \frac{(\mathbf{y}_2)^{\bar{\alpha}_2}}{\bar{\alpha}_2!} \ldots \frac{(\mathbf{y}_k)^{\bar{\alpha}_k}}{\bar{\alpha}_k!}$$

where$(\mathbf{y}_j)^{\bar{\alpha}_j} = (y_j, y_{j,1}, \ldots, y_{j,n_j})^{(\alpha_j, \alpha_{j,1}, \ldots, \alpha_{j,n_j})} = \prod_{p_j=0}^{n_j} (y_{j,p_j})^{\alpha_{j,p_j}}$ for $j = 1, \ldots, k$

Dual of $\mathbf{u}_1^{\delta_1}\mathbf{u}_2^{\delta_2}\ldots\mathbf{u}_k{}^{\delta_k}$ is the evaluation $\mathbf{e}_{\mathbf{u}}$ at $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k)$.

Thus if $T = \sum_{i=1}^{r} \omega_i \mathbf{u}_{i,1}^{\delta_1}\mathbf{u}_{i,2}^{\delta_2}\ldots\mathbf{u}_{i,k}{}^{\delta_k}$, then T^* coincides with the weighted sum of evaluations $T^* = \sum_i^r \omega_i\, \mathbf{e}_{\mathbf{u}_i}$ on $R_{\delta_1,\delta_2,\ldots,\delta_k}$.

Multi-Linear Tensor Decomposition Problem

$A_1, A_2 \mid R_{1,1,\ldots,1}$

- $H_0 = H_{T^*}^{A_1,A_2} = [t_{\alpha+\beta}]_{\alpha\in A_1 \beta\in A_2}$
- $H_{1,i_1} = H_{x_{1,i_1}\star T^*}^{A_1,A_2} = H_{T^*}^{x_{1,i_1} A_1,A_2} = [t_{\alpha+\beta}]_{\alpha\in x_{1,i_1}\star A_1, \beta\in A_2}$
- $\widehat{H}_0 = \sum_{i_1=0}^{n_1} \lambda_{i_1} H_{1,i_1}$

- Truncated Singular Value Decomposition: H_0 ,$H_{x_{1,i_1}}$ and $\widehat{H}_0 \rightsquigarrow H_0^r$,$H^r_{x_{1,i_1}}$ and $\widehat{H}^r_0$
-

$$H^r_{x_{1,i_1}} = (M^{{U_r}^{\mathbf{H}}}_{x_{1,i_1}})^{\mathbf{T}} H^r_0 = H^r_0 M^{\overline{V}_r}_{x_{1,i_1}}$$

where $M^{{U_r}^{\mathbf{H}}}_{x_{1,i_1}}$ (resp. $M^{\overline{V}_r}_{x_{1,i_1}}$) is the multiplication matrix by x_{1,i_1} in the basis ${U_r}^{\mathbf{H}}$ (resp. $\overline{V}_r$) and $M^{\overline{V}_r}_{x_{1,i_1}}$ is the multiplication matrix by x_{1,i_1} in the basis $\overline{V}_r$.

- By linearity: $\widehat{H}^r_0 = \sum_{i_1=0}^{n_2} \lambda_{i_1} H^r_{x_{1,i_1}} = H^r_0 \sum_{i_1=0}^{n_2} \lambda_{i_1} M^{\overline{V}_r}_{x_{1,i_1}} = H^r_0 M^{\overline{V}_r}_{\lambda(\mathbf{x_1})} \Rightarrow$

$$(\widehat{H}^r_0)^{-1} = (M^{\overline{V}_r}_{\lambda(\mathbf{x_1})})^{-1} (H^r_0)^{-1}$$

$$(\widehat{H}^r_0)^{-1} H^r_{x_{1,i_1}} = (M^{\overline{V}_r}_{\lambda(\mathbf{x_1})})^{-1} M^{\overline{V}_r}_{x_{1,i_1}} = M^{\overline{V}_r}_{(x_{1,i_1}/\lambda(\mathbf{x_1}))}$$

We compute the eigenvalues and the eigenvectors of the multiplication matrices $M^{\overline{V}_r}_{(x_{1,i_1}/\lambda(\mathbf{x_1}))}$ in order to obtain the weights and the points of the decomposition.

Multiplication Operators in the orthogonal basis

Proposition: Let $\sigma = \sum_{i=1}^{r'} \omega_i \mathbf{e}_{\xi_i}$ with $\omega_i \in \mathbb{C}$, $\xi_i \in \mathbb{C}^n$ are simple.
If rank $H_\sigma^{A_1, A_2} = r$,

- The matrix $M_{x_{1,i_1}}^{\overline{V}_r}$ associated to the multiplication operator by y_i in the basis $\overline{V}_r$ of $\mathcal{A}_\sigma$ is $M_{x_{1,i_1}}^{\overline{V}_r} = S_r{}^{-1}\, U_r{}^{\mathsf{H}}\, H_{x_{1,i_1}\star\sigma}^{\overline{A}_1, \overline{A}_2}\, \overline{V}_r$ $i = 1, \ldots, n$.

Algorithm 5.1: Decomposition of Tri-Linear Tensor with constant weights

the moments $(t_{i,j,k})_{\substack{0\le i\le n_1\\ 0\le j\le n_2\\ 0\le k\le n_3}}$ of σ.

1. Compute the monomial sets $A_1 = (x_i)_{0\le i\le n_1}$ and $A_2 = (z_k)_{0\le k\le n_3}$ and substitute the x_0, y_0 and z_0 by 1 to define $\overline{A}_1$ and $\overline{A}_2$.
2. Compute the truncated Hankel matrix $H_\sigma^{\overline{A}_1,\overline{A}_2}$ for the monomial sets $\overline{A}_1$ and $\overline{A}_2$.
3. Compute the singular value decomposition of $H_\sigma^{\overline{A}_1,\overline{A}_2} = USV^{\mathsf{T}}$ where $\overline{A}_1 = \langle 1, x_1, \ldots, x_{n_1}\rangle$ and $\overline{A}_2 = \langle 1, z_1, \ldots, z_{n_3}\rangle$ with singular values $s_1 \ge s_2 \ge \cdots \ge s_r \ge 0$.
4. Determine its numerical rank, that is, the largest integer r such that $\frac{s_r}{s_1} \ge \epsilon$.
5. Form the multiplication matrices by x_{2,j_2} in the basis $\overline{V}_r$, $M_{x_{2,j_2}}^{\overline{V}_r} = S_r^{-1} U_r^{\mathsf{H}} H_{x_{2,j_2}\star\sigma}^{\overline{A}_1,\overline{A}_2} \overline{V}_r$ where $H_{x_{2,j_2}\star\sigma}^{\overline{A}_1,\overline{A}_2}$ is the Hankel matrix associated to $x_{2,j_2}\star\sigma$ for $j = 1, \ldots, n_2$.

6. Compute the eigenvectors $\mathbf{v}_p$ of $\sum_{j=1}^{n_2} l_j M_{x_{2,j_2}}^{\overline{V}_r}$ such that $|l_j| \leq 1, j = 1, \ldots, n_2$ and for each $p = 1, \ldots, r$ do the following:

- The $y's$ coordinates of the $\mathbf{u}_p$ are the eigenvalues of the multiplication matrices by x_{2,j_2}. Use the formula $M_{x_{2,j_2}}^{\overline{V}_r} \mathbf{v}_p = b_{p,j} \mathbf{v}_p$ for $p = 1, \ldots, r$ and $j = 1, \ldots, n_2$ and deduce the $b_{p,j}$.
- Write the matrix $H_\sigma^{\overline{A}_1, \overline{A}_2}$ in the basis of interpolation polynomials(ie. the eigenvectors $\mathbf{v}_p$) and use the corresponding matrix $\mathcal{T} = [\sigma(x_{3,i_3} \mathbf{v}_j)]_{\substack{1 \leq i \leq n_3 \\ 1 \leq j \leq r}}$ to compute the $z's$ coordinates. Divide the $k^{\mathbf{th}}$ row on the first row of the matrix $\mathcal{T}$ to obtain the values of $c_{p,k}$ for $p = 1, \ldots, r$ and $k = 1, \ldots, n_3$.
- The $x's$ coordinates of $\underline{\mathrm{u}}_p$ are computed using the eigenvectors of the transpose of the matrix $M_{x_{2,j_2}}^{\overline{V}_r}$. They -are up to scalar- the evaluations, they are represented by vectors of the form $\mathbf{v}_p^* = \mu_p[1, a_{p,1}, \ldots, a_{p,n_1}]$. Compute $\mathbf{v}_p^*$ as the p^{th} column of the transpose of the inverse of the matrix $V = [v_1, \ldots, v_r]$ for $p = 1, \ldots, r$ and deduce $a_{p,i} = \frac{\mathbf{v}_p^*[i+1]}{\mathbf{v}_p^*[1]}$ for $p = 1, \ldots, r$ and $i = 1, \ldots, n_1$.
- Compute $\omega_p = \frac{\langle \sigma | \mathbf{v}_p \rangle}{\mathbf{v}_p(\mathbf{u}_p)}$.

Ex3:Decomposition of Real coefficients Tensor

$\mathbf{x} = (x_0, x_1, x_2, x_3), \mathbf{y} = (y_0, y_1, y_2, y_3), \mathbf{x} = (z_0, z_1, z_2, z_3).$

$P(\mathbf{x}, \mathbf{y}, \mathbf{z}) = 0.0264x_0y_0z_0 - 0.0207x_0y_0z_1 - 0.0065x_0y_0z_2 - 0.0208x_0y_0z_3 - 0.0512x_0y_1z_0 + 0.0315x_0y_1z_1 - 0.0353x_0y_1z_2 + .1331x_0y_1z_3 + 0.0056x_0y_2z_0 - 0.0194x_0y_2z_1 + 0.0256x_0y_2z_2 - .1072x_0y_2z_3 - 0.0249x_0y_3z_0 + 0.0231x_0y_3z_1 + 0.0071x_0y_3z_2 + 0.0261x_0y_3z_3 - 0.0017x_1y_0z_0 - 0.0196x_1y_0z_1 - 0.0173x_1y_0z_2 - 0.0092x_1y_0z_3 + .1287x_1y_1z_0 + 0.0253x_1y_1z_1 + 0.04x_1y_1z_2 + 0.0464x_1y_1z_3 - 0.0149x_1y_2z_0 + 0.0284x_1y_2z_1 + 0.0117x_1y_2z_2 + 0.0347x_1y_2z_3 - 0.0101x_1y_3z_0 + 0.0146x_1y_3z_1 + 0.0131x_1y_3z_2 - 0.0005x_1y_3z_3 - 0.0199x_2y_0z_0 + 0.0202x_2y_0z_1 + 0.0025x_2y_0z_2 + 0.0329x_2y_0z_3 + 0.0528x_2y_1z_0 - 0.0661x_2y_1z_1 + 0.0352x_2y_1z_2 - .2197x_2y_1z_3 - 0.0268x_2y_2z_0 + 0.0411x_2y_2z_1 - 0.0288x_2y_2z_2 + .1569x_2y_2z_3 + 0.0209x_2y_3z_0 - 0.022x_2y_3z_1 - 0.002x_2y_3z_2 - 0.038x_2y_3z_3 + 0.0267x_3y_0z_0 - 0.0225x_3y_0z_1 - 0.0181x_3y_0z_2 + 0.0028x_3y_0z_3 + 0.0257x_3y_1z_0 - 0.0267x_3y_1z_1 - 0.005x_3y_1z_2 - 0.0399x_3y_1z_3 - 0.038x_3y_2z_0 + 0.0359x_3y_2z_1 + 0.0157x_3y_2z_2 + 0.03x_3y_2z_3 - 0.0268x_3y_3z_0 + 0.0225x_3y_3z_1 + 0.0185x_3y_3z_2 - 0.0036x_3y_3z_3$

$$H_{\sigma}^{\overline{A}_1,\overline{A}_2} = USV^{T} = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc|c} 1 & x_1 & x_2 & x_3 & \\ \end{array}$$

$$\begin{array}{c|c} \begin{array}{cccc} 1 & x_1 & x_2 & x_3 \end{array} & \\ \begin{bmatrix} t_{0,0,0} & t_{1,0,0} & t_{2,0,0} & t_{3,0,0} \\ t_{0,0,1} & t_{1,0,1} & t_{2,0,1} & t_{3,0,1} \\ t_{0,0,2} & t_{1,0,2} & t_{2,0,2} & t_{3,0,2} \\ t_{0,0,3} & t_{1,0,3} & t_{2,0,3} & t_{3,0,3} \end{bmatrix} & \begin{array}{c} 1 \\ z_1 \\ z_2 \\ z_3 \end{array} \end{array}$$

$$= \begin{array}{cc} \begin{array}{cccc} 1 & x_1 & x_2 & x_3 \end{array} & \\ \begin{bmatrix} 0.0264 & -0.0017 & -0.0199 & 0.0267 \\ -0.0207 & 0.0196 & 0.0202 & -0.0225 \\ -0.0065 & -0.0173 & 0.0025 & -0.0181 \\ -0.0208 & -0.0092 & 0.0329 & 0.0028 \end{bmatrix} & \begin{array}{c} 1 \\ z_1 \\ z_2 \\ z_3 \end{array} \end{array}$$

$$S = \begin{bmatrix} 0.0681 & 0 & 0 & 0 \\ 0 & 0.0284 & 0 & 0 \\ 0 & 0 & 0.0199 & 0 \\ 0 & 0 & 0 & 3.3112 * 10^{-12} \end{bmatrix}$$

epsilon$= 10^{-10}$ **r**$= 3$

$$
H^{\overline{A}_1,\overline{A}_2}_{y_i\sigma} = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}
\begin{array}{cccc|c}
y_i 1 & y_i x_1 & y_i x_2 & y_i x_3 & \\
\end{array}
$$

$$
H^{\overline{A}_1,\overline{A}_2}_{y_i\sigma} =
\begin{array}{c@{}c}
\begin{array}{cccc} y_i 1 & y_i x_1 & y_i x_2 & y_i x_3 \end{array} & \\
\begin{bmatrix}
t_{0,0,0} & t_{1,0,0} & t_{2,0,0} & t_{3,0,0} \\
t_{0,0,1} & t_{1,0,1} & t_{2,0,1} & t_{3,0,1} \\
t_{0,0,2} & t_{1,0,2} & t_{2,0,2} & t_{3,0,2} \\
t_{0,0,3} & t_{1,0,3} & t_{2,0,3} & t_{3,0,3}
\end{bmatrix} & \begin{array}{c} 1 \\ z_1 \\ z_2 \\ z_3 \end{array}
\end{array}
$$

for $i = 1, \ldots, n$

$$H_{y_1\sigma}^{\overline{A}_1,\overline{A}_2} = \begin{bmatrix} -0.0512 & 0.1287 & ,0.0528 & 0.0257 \\ 0.0315 & 0.0253 & -0.0661 & -0.0267 \\ -0.0353 & 0.04 & 0.0352 & -0.005 \\ 0.1331 & 0.0464 & -0.2197 & -0.0399 \end{bmatrix}$$

$$H_{y_2\sigma}^{\overline{A}_1,\overline{A}_2} = \begin{bmatrix} 0.0056 & -0.0149 & -0.0268 & -0.038 \\ -0.0194 & 0.0284 & 0.0411 & 0.0359 \\ 0.0256 & 0.0117 & -0.0288 & 0.0157 \\ -0.1072 & 0.0347 & 0.1569 & 0.03 \end{bmatrix}$$

$$H_{y_3\sigma}^{\overline{A}_1,\overline{A}_2} = \begin{bmatrix} -0.0249 & -0.0101 & 0.0209 & -0.0268 \\ 0.0231 & 0.0146 & -0.022 & 0.0225 \\ 0.0071 & 0.0131 & -0.002 & 0.0185 \\ 0.0261 & -0.0005 & -0.038 & -0.0036 \end{bmatrix}$$

Points =

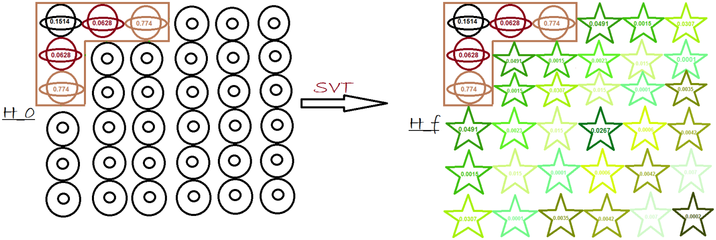
$\mathbf{u}_i$	$\mathbf{u}_1$	$\mathbf{u}_2$	$\mathbf{u}_3$
$a_{i,0}$	1	1	1
$a_{i,1}$	0.9535	-0.2373	-4.7475
$a_{i,2}$	-0.6526	-1.4691	-0.1451
$a_{i,3}$	1.6178	-0.2519	0.0074
$b_{i,0}$	1	1	1
$b_{i,1}$	0.5811	-6.9117	-6.4635
$b_{i,2}$	-1.1104	4.897	-0.5801
$b_{i,3}$	-1.0127	-1.1502	-0.2973
$c_{i,0}$	1	1	1
$c_{i,1}$	-0.8677	-1.3065	0.4138
$c_{i,2}$	-0.6216	0.5333	0.3747
$c_{i,3}$	-0.0957	-3.975	0.7576

Weights =

ω_1	0.0173
ω_2	0.0055
ω_3	0.0035

Completion of Hankel Matrices

Numerical Example



Completion of structured matrices

Minimisation Problem

$$\begin{aligned} \underset{X}{\text{minimize}} \quad & \operatorname{rank}(X) \\ \text{subject to} \quad & \mathcal{P}_\Omega(X) = \mathcal{P}_\Omega(Y) \end{aligned}$$

Convex Relaxation Problem

$$\begin{aligned} \underset{X}{\text{minimize}} \quad & \|X\|_* \\ \text{subject to} \quad & \mathcal{P}_\Omega(X) = \mathcal{P}_\Omega(Y) \end{aligned}$$

For $\tau > 0$

$$\begin{aligned} \underset{X}{\text{minimize}} \quad & \tau \|X\|_* + \frac{1}{2} \|X\|_F \\ \text{subject to} \quad & \mathcal{P}_\Omega(X) = \mathcal{P}_\Omega(Y) \end{aligned}$$

Completion of structured matrices

SVT

Generalized Problem

$$\begin{aligned} \underset{X}{\text{minimize}} \quad & \tau \|X\|_* + \frac{1}{2}\|X\|_F \\ \text{subject to} \quad & \mathcal{A}(X) = b \end{aligned}$$

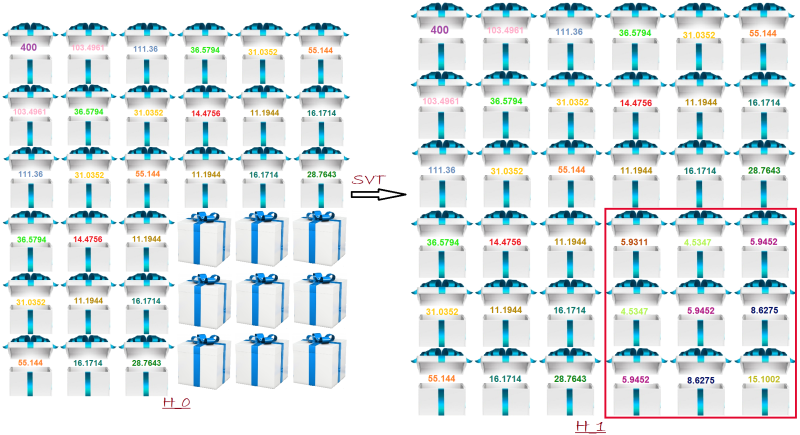
Usawa's Algorithm $b = \mathcal{A}(X_0), y^0 = b * k_0 * \delta$

$$\begin{cases} X^k = \mathbb{D}_\tau(\mathcal{A}^*(y^{k-1})) \\ y^k = y^{k-1} + \delta_k(b - \mathcal{A}(X^k)) \end{cases}$$

Usawa's Algorithm

$$\begin{cases} \mathcal{A}(X, L) = [< L[i], X >]_{i=1}^n \\ \mathcal{A}^*(y, L) = \sum_{i=1}^{nops(L)} y[i] * L[i] \\ \mathcal{A}^*\mathcal{A} = \mathcal{P}_\Omega \\ \mathcal{A}(Y) = b \end{cases}$$

Newton Influence



Points =

$\mathbf{u}_i$	$u_{i,0}$	$u_{i,1}$	$u_{i,2}$
$\mathbf{u}_1$	1.0	0.4404	0.5359
$\mathbf{u}_2$	1.0	0.1324	0.4976
$\mathbf{u}_3$	1.0	0.6729	-0.0390
$\mathbf{u}_4$	1.0	0.1437	0.0410

Weights =

ω_1	112.805
ω_2	87.1916
ω_3	174.075
ω_4	25.928

Points =

$\mathbf{u}_i$	$u_{i,0}$	$u_{i,1}$	$u_{i,2}$
$\mathbf{u}_1$	1.0	0.4310	0.5427
$\mathbf{u}_2$	1.0	0.1359	0.5034
$\mathbf{u}_3$	1.0	0.3952	0.0094
$\mathbf{u}_4$	1.0	0.0723	0.0579

Weights =

ω_1	100.0
ω_2	100.0
ω_3	100.0
ω_4	100.0

The error on the series before and after completion and Newton step is $1.6377e^{-14}$

Newton Method

⋆ A : finite subset of $\mathbb{N}^n$
⋆ $\omega_i, \xi_{i,j}$ are variables
⋆ $\Xi = (\xi_{i,j})_{1\leq i\leq r, 0\leq j\leq n}$ set of variables / $\xi_{i,0} = \omega_i$ for $i = 1, \ldots, r$
⋆ $I = [1, r] \times [0, n] = \{(i,j) \mid 1 \leq i \leq r, 0 \leq j \leq n\}$ be the indices of the variables and $N = (n+1)\, r = |I|$
⋆ $\alpha \in A$ $F_\alpha(\Xi) = \sum_{i=1} \omega_i \xi_i^\alpha - \tilde{\sigma}_\alpha$ be the error function for the moment $\tilde{\sigma}_\alpha$
⋆ $F(\Xi) = (F_\alpha(\Xi))_{\alpha \in A}$ the vector of these error functions

minimize the distance

$$E(\Xi) = \frac{1}{2} \sum_{\alpha \in A} |F_\alpha(\Xi)|^2 = \frac{1}{2} \|F(\Xi)\|^2$$

Newton Method

$\star$ $M(\Xi_i) = [\omega_i \xi_i^{\alpha}]_{\alpha \in A} \rightsquigarrow V(\Xi) = \left(\partial_{(i,j)} M(\Xi_i)\right)_{(i,j)\in I}$ the $|A| \times N$ Vandermonde-like matrix.

Gradient of $E(\Xi)$

$$\nabla E(\Xi) = (< \partial_{(i,j)} M(\Xi_i), F(\Xi) >)_{(i,j)\in I} = V(\Xi)^T F(\Xi)$$

System

$$\nabla E(\Xi) = 0$$

Jacobien:

$$\begin{aligned} J_{\Xi}(\nabla E) &= < \partial_{(i,j)} M(\Xi_j), \partial_{(i',j')} M(\Xi_{j'}) > + \left(< \partial_{(i,j)} \partial_{(i',j')} M(\Xi_i), F(\Xi) >\right)_{(i,j)\in I, (i',j')\in I} \\ &= V(\Xi)^{\mathsf{T}} V(\Xi) + \left(< \partial_{(i,j)} \partial_{(i',j')} M(\Xi_i), F(\Xi) >\right)_{(i,j)\in I, (i',j')\in I} . \end{aligned}$$

$\partial_{(i,j)} \partial_{(i',j')} M(\Xi_i) = 0$ if $i \neq i'$

Newton iteration:

$$\Xi_{n+1} = \Xi_n - J_{\Xi}(\nabla E)^{-1} \nabla E(\Xi_n).$$

Newton Influence on series'decomposition

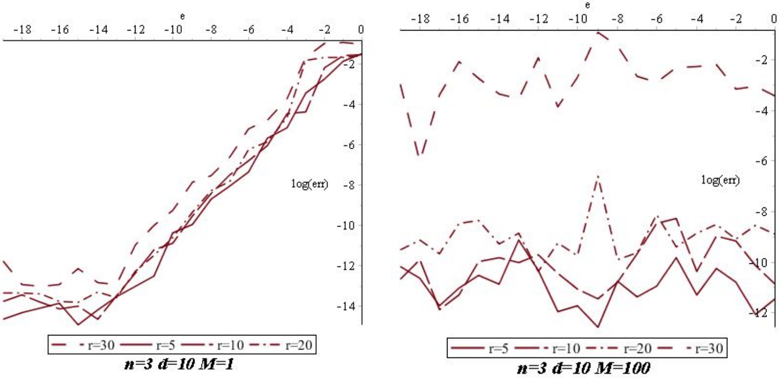


Figure: The evolution of the error in terms of the perturbation $\epsilon = 10^{(-e)}$ on the moments of the form $\sigma_\alpha + \epsilon(p_\alpha + iq_\alpha)$ with amplitude M for different values of r with 5 Newton iterations.

Conclusions and Perspectives

⋆ Decomposition algorithm of the low rank the Hankel operator.

⋆ A basis of $\mathcal{A}_\sigma$ is computed from the Singular Value Decomposition of a sub-matrix.

⋆ An explicit formula for the weights in terms of the eigenvectors of multiplication matrices.

⋆ Rescaling technique to improve the numerical quality of the reconstruction of frequencies.

⋆ Newton iteration

⋆ Singular Value Thresholding Algorithm for the completion of low rank Hankel matrix

- New technique which solves the numerical instability of the decomposition problem when the multiplicities of points are more than one.
- Other applications: the phylogenetic trees and image processing..

Merci

Thank you