Simple Forms and Rational Solutions of Pseudo-Linear systems

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Introduction

Objectives

1. A generic and direct algorithm for computing simple forms of pseudo-linear systems of the form:

$$A\delta(y) + B\phi(y) = 0.$$

2. Introduce an alternative, based on simple forms, to compute the bounds of the numerator and the denominator of all rational solutions of a q-difference system:

$$y(qx) = M(x)y(x).$$

Motivation

- Simple forms was introduced by Barkatou 1999 as an alternative to super-irreducibles forms.
- They give useful information:
  - Local data: nature of a singularity, indicial equation, regular solutions,...
  - Global solutions: polynomial, rational, hyperexponential, ...
- Methods for computing simple forms of pseudo-linear systems have been developed before. They require applying super-reduction algorithms first, which are costly.
- Direct algorithms to compute simple forms appeared in the differential and difference case [Barkatou, Cluzeau, El Bacha 2011, 2018].

Pseudo-linear systems

- Let C a field of characteristic 0, and $K = C((t))$ equipped with the t-adic valuation ν.
- ϕ a C-automorphism of K satisfying $\nu(\phi(a)) = \nu(a) \quad \forall a \in K$.
- δ is ϕ-derivation, i.e: $\delta(ab) = \phi(a)\delta(b) + \delta(a)b \quad \forall a, b \in K$.
- The constants of K are elements c satisfying $\phi(c) = c$ and $\delta(c) = 0$.

Remark: If $\phi \neq \mathrm{id}_K$ then $\delta = \gamma(\phi - \mathrm{id}_K)$ for some $\gamma \in K^*$.

Definition

A fully integrable pseudo-linear system of size n over K is a system of the form

$$A\delta(y) + B\phi(y) = 0$$

where $A, B \in \mathcal{M}_n(C[[t]])$, $\det(A) \neq 0$ and $\det(-B + \gamma A) \neq 0$.

Familiar examples

- Differential systems:

$$K = \mathbb{C}((x)), \quad \phi = \mathrm{id}_K, \quad \delta = \frac{d}{dx}.$$

- Difference systems:

$$K = \mathbb{C}((x^{-1})), \quad \phi(x) = x + 1, \quad \delta = \mathrm{id}_K - \phi.$$

- q-difference systems:

$$K = \mathbb{C}((x)), \quad \phi(x) = qx, \quad q \in \mathbb{C}^*, \quad \delta = \phi - \mathrm{id}_K.$$

Equivalent pseudo-linear systems

To any pseudo-linear system we associate the operator

$$L = A\delta + B\phi.$$

The system can be written $L(y) = 0$.

Definition

Two operators $L = A\delta + B\phi$ and $L' = A'\delta + B'\phi$ are said to be equivalent if $\exists\ S, T \in \mathrm{GL}_n(K)$ such that $L' = S\,L\,T$, that is:

$$A' = S\,A\,T, \qquad B' = S\,A\,\delta(T) + S\,B\,\phi(T).$$

Two pseudo-linear systems $L(y) = 0$ and $L'(y) = 0$ are equivalent if the operators L and L' are equivalent.

Simple forms

- $A = A_0 + tA_1 + t^2 A_2 + \ldots$
- $B = B_0 + tB_1 + t^2 B_2 + \ldots$
- Leading pencil: $L_\lambda = A_0 \, \lambda + B_0 \in \mathcal{M}_n(C[\lambda])$.

Definition

We say that a pseudo-linear system is simple if $\det(L_\lambda) \neq 0$.

Remark: Non-simple pseudo-linear system $\implies \nu(\det(A)) > 0$.

Computing simple forms

The method

- The method is a generalization of the ideas in the differential and difference case [Bar-El Bacha 2011, Bar-Clu-El Bacha 2018].
- Two key points: ϕ preserves the valuation, and the notion of equivalence.

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- Two key points: ϕ preserves the valuation, and the notion of equivalence.
- Compute iteratively equivalent pseudo-linear systems

$$A^{(i)}\,\delta(y) + B^{(i)}\,\phi(y) = 0, \; i \geq 1$$

satisfying $\nu(\det(A^{(i+1)})) < \nu(\det(A^{(i)}))$.

- We are sure we reach a simple system.

The method

- $A_0 = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \quad r = \mathrm{rank}(A_0), \quad B_0 = \begin{pmatrix} B_0^{11} & B_0^{12} \\ B_0^{21} & B_0^{22} \end{pmatrix}$
- The matrix L_λ is written as

$$L_\lambda = \begin{pmatrix} I_r\lambda + B_0^{11} & B_0^{12} \\ B_0^{21} & B_0^{22} \end{pmatrix}.$$

- The rows of $(B_0^{21} \quad B_0^{22})$ are called the λ-free rows.

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- The rows of $(B_0^{21} \quad B_0^{22})$ are called the λ-free rows.
- Two cases arise:
  1. $\mathrm{rank}(B_0^{21} \quad B_0^{22}) < n - r \implies$ we can decrease $\nu(\det(A))$.
  2. $\mathrm{rank}(B_0^{21} \quad B_0^{22}) = n - r \implies$ we can reduce to case 1

     without increasing $\nu(\det(A))$.

Case 1: λ-free rows are linearly dependent

- If we have $\mathrm{rank}(B_0^{21} \quad B_0^{22}) < n - r$, then we can always construct an invertible matrix $S = DC$ in K such that the equivalent operator $\widehat{L} = S\,L = \widehat{A}\,\delta + \widehat{B}\,\phi$ satisfies

$$\nu(\det(\widehat{A})) < \nu(\det(A)).$$

- Here $D \in \mathrm{GL}_n(K)$ is given by

$$D = \mathrm{diag}(1, \dots, 1, t^{-\mu}, 1, \dots, 1), \quad \text{for some} \quad \mu > 0.$$

- C is a constant matrix in $\mathcal{M}_n(C)$ which transforms the $(r+i)$th row of L_λ into zeros, for some position $r+i$.

Case 1: λ-free rows are linearly dependent

Example

Let $K = C((x))$, $\phi(x) = qx$, $q \in C^*$, and $\delta = \phi - \mathrm{id}_K$. Consider the q-difference operator:

$$L = \underbrace{\begin{bmatrix} 1+x & \frac{x}{q^2} \\ 0 & x \end{bmatrix}}_{A} \delta + \begin{bmatrix} (1+x(q+1))(q-1) & \frac{x(q^2-1)}{q^2} \\ (xq+1)xq & \frac{(-q+1+xq)x}{q} \end{bmatrix} \phi$$

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad L_\lambda = \begin{bmatrix} \lambda + q - 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Case 1: λ-free rows are linearly dependent

- $C = I_2$ and $D = \mathrm{diag}(1, x^{-1})$.
- Multiplication of L on the left by D:

$$\widehat{L} = \underbrace{\begin{bmatrix} x+1 & \frac{x}{q^2} \\ 0 & 1 \end{bmatrix}}_{\widehat{A}} \delta + \begin{bmatrix} (q-1)(1+x(q+1)) & \frac{x(q^2-1)}{q^2} \\ (qx+1)q & \frac{1+(x-1)q}{q} \end{bmatrix} \phi$$

- $\nu(\det(\widehat{A})) = 0 < \nu(\det(A)) = 1.$
- The leading pencil of $\widehat{L}$:

$$\widehat{L}_\lambda = \begin{bmatrix} \lambda + q - 1 & 0 \\ q & \frac{1+(\lambda-1)q}{q} \end{bmatrix}$$

which is regular.

Case 2: λ-free rows are linearly independent

- If we have $\mathrm{rank}(B_0^{21} \quad B_0^{22}) = n - r$, then we can always construct two invertible matrices $S = DC$ and T in K such that the λ-free rows of the equivalent operator $\widehat{L} = SLT$ are linearly dependent.
- Moreover if we note $\widehat{L} = \widehat{A}\,\delta + \widehat{B}\,\phi$, then we have

$$\nu(\det(\widehat{A})) = \nu(\det(A)).$$

- C is a constant matrix.
- $D \in \mathrm{GL}_n(K)$ is given by

$$D = \mathrm{diag}(t^{-1} I_p, I_{r-p}, I_{n-r}), \quad \text{for some} \quad p \geq 0.$$

- $T = D^{-1}$.

Case 2: λ-free rows are linearly independent

Example

Let $K = C((x))$, $\phi(x) = qx$, $q \in C^*$, and $\delta = \phi - \mathrm{id}_K$. Consider the q-difference operator:

$$L = \underbrace{\begin{bmatrix} x & 0 \\ 0 & x+1 \end{bmatrix}}_{A} \delta + \begin{bmatrix} -x & qx+1 \\ -\frac{x}{q} & \frac{(q^3-q^2+q)x+1}{q^2} \end{bmatrix} \phi$$

- The operator L is not simple and we have

$$\nu(\det(A)) = 1.$$

Case 2: λ-free rows are linearly independent

- The leading matrix pencil of the operator is:

$$L_\lambda = \begin{bmatrix} \lambda + q^{-2} & 0 \\ 1 & 0 \end{bmatrix}.$$

- Multiply on the left by $S = DC$ and on the right by $T = D^{-1}$ where:

$$C = \begin{bmatrix} 1 & -\frac{1}{q^2} \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \mathrm{diag}(x^{-1}, 1)$$

Case 2: λ-free rows are linearly independent

- This yield an equivalent operator $\widehat{L}$ such that:

$$\widehat{L} = \underbrace{\begin{bmatrix} 1+x & \frac{x}{q^2} \\ 0 & x \end{bmatrix}}_{\widehat{A}} \delta + \begin{bmatrix} q-1+xq^2-x & \frac{x\left(q^2-1\right)}{q^2} \\ \left(xq+1\right)xq & \frac{(-q+1+xq)x}{q} \end{bmatrix} \phi$$

with

$$\widehat{L}_\lambda = \begin{bmatrix} \lambda+q-1 & 0 \\ 0 & 0 \end{bmatrix}.$$

- Moreover $\nu(\det(\widehat{A})) = \nu(\det(A)) = 1$.

Complexity estimate

Proposition

Consider a pseudo-linear system $A\delta(y) + B\phi(y) = 0$. *Let* $d = \nu(\det(A))$ *and suppose that the t-adic expansions of A and B are known up to an order* $k \geq d$. *Then Algorithm* `SimpleForm` *computes a simple form using at most*

$$\mathcal{O}(n^{\omega+1}\, d + k\, n^3\, d)$$

arithmetic operations in the field C, *where* ω *denotes the linear algebra exponent.*

Implementation

- Algorithm `SimpleForm` is available at http://www.unilim.fr/pages_perso/ali.el-hajj/Implementations.html.
- For the previous example, the user must first define the matrices A and B.
- Then the user should define:

```
> PhiAction:= proc(M,x) return subs(x=q*x,M) end;
> DeltaAction:= proc(M,x) return PhiAction(M,x)-M
end;
> t:=x
```

Implementation

Then the user should run:

```
> Simple_Form(A,B,DeltaAction,PhiAction,x,t);
```

$$\left[\begin{array}{cc} \frac{\alpha+x}{\alpha} & \frac{x}{\alpha^2 q^2} \\ 0 & 1 \end{array}\right], \left[\begin{array}{cc} \frac{xq^2}{\alpha}+q-1-\frac{x}{\alpha} & \frac{x}{\alpha^2}-\frac{x}{\alpha^2 q^2} \\ xq^2+\alpha\, q & -1+\frac{x}{\alpha}+q^{-1} \end{array}\right]$$

$$\left[\begin{array}{cc} -\frac{1}{\alpha\, xq^2} & \frac{1}{\alpha\, x} \\ x^{-1} & 0 \end{array}\right], \left[\begin{array}{cc} 0 & 1 \\ x & \frac{x}{\alpha\, q^2} \end{array}\right], \frac{(\lambda+q-1)(\lambda\, q-q+1)}{q}, \quad 0.040$$

Application: Rational solutions of q-difference systems

Problem

- Let $K = C(x)$. Compute all rational solutions of

$$y(qx) = M(x)y(x), \qquad M \in \mathrm{GL}_n(K), \quad q \in C/\{0,1\}. \quad (1)$$

- Define the automorphism ϕ by $\phi(x) = qx \quad \forall x \in K$.
- An algorithm to solve this problem is already developed by Abramov 2002.
  1. Compute a universal denominator $u = u_1^{d_1} \dots u_k^{d_k}$.
  2. Let $y = z/u$. Compute polynomial solutions in z, of degree bound d, of a system of the same type as (1).
- The problem is to find the degree bounds $d_1, \dots, d_k$ and d.

Computing the degree bounds

- If the singularity is not fixed by $\phi \implies$ the corresponding d_i is obtained by gcd's calculations.
- for $x = 0$, the fixed singularity by ϕ, the corresponding d_i is obtained by inspecting the integer roots of the indicial polynomial $\varphi(\lambda)$ at this singularity.
- The degree bound d of the polynomial solutions is again obtained by inspecting $\varphi(\lambda)$ at ∞.
- The problem is how to get efficiently $\varphi(\lambda)$.

Computing $\varphi(\lambda)$

- Existing algorithm: EG-eliminations [Abramov 1999].
  It provides extra solutions.

- Alternative algorithm: `SimpleForm`.

- If the system is simple $\implies$ $\varphi(\lambda)$ reads

$$\varphi(\lambda) = \det((1 - q^{\lambda})A_0 + q^{\lambda}B_0).$$

Example

Consider the q-difference system

$$\phi(y) = \begin{bmatrix} \frac{q^2+1}{q} & -\frac{(\alpha+x)q}{x} \\ \frac{x}{qx+\alpha} & 0 \end{bmatrix} y, \qquad \alpha \neq 0.$$

We write the system as $A\delta(y) + B\phi(y) = 0$ where $\delta = \phi - \mathrm{id}_K$,

$$A = \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -x & qx+\alpha \\ -\frac{x}{(\alpha+x)q} & \frac{\left(q^3-q^2+q\right)x+\alpha}{(\alpha+x)q^2} \end{bmatrix}.$$

Example

- Universal denominator $u = x^{d_1}(x+\alpha)^{d_2}$
- by gcd's calculations $\implies$ $d_2 = 1$.
- The system is not simple at 0 $\implies$ apply `SimpleForm` to get

$$\varphi(\lambda) = q^{2\lambda} - q^{\lambda-1} - q^{\lambda+1} + 1.$$

- -1 is the smallest integer root of $\varphi(\lambda)$ $\implies$ $d_1 = |-1| = 1$.

Example

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- by gcd's calculations $\Longrightarrow$ $d_2 = 1$.
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$$\varphi(\lambda) = q^{2\,\lambda} - q^{\lambda-1} - q^{\lambda+1} + 1.$$

- -1 is the smallest integer root of $\varphi(\lambda)$ $\Longrightarrow$ $d_1 = |-1| = 1$.
- Let $y = z/u$, we get an equivalent system which is already simple at ∞ with

$$\varphi(\lambda) = 1 - q^{-\lambda-1} - q^{-\lambda-3} + q^{-2\,\lambda-4}.$$

- -3 is the smallest integer root of $\varphi(\lambda)$ $\Longrightarrow$ the degree bound d of the polynomial solutions is $d = |-3| = 3$.

Conclusions

- We developed a unified and direct algorithm to compute a simple form of pseudo-linear systems.
- This algorithm is very helpful in the local study of these systems.
- We proved that this algorithm could be used as an alternative to previous algorithm for computing rational solutions of q-difference systems.
- Our implementation allows to obtain a new efficient implementation for computing rational solutions of pseudo-linear systems.
- Moreover, this implementation can be extended to handle a set of several pseudo-linear systems.

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Thank you !