Reduction operators and completion of linear rewriting systems

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February 8, 2019

Plan

I. Motivations

- ▷ Computational problems in algebra and rewriting theory
- ▷ Termination, confluence and Gröbner bases

II. Reduction operators

- ▷ Reduction operators and linear rewriting systems
- ▷ Lattice structure of reduction operators

III. Confluence and completion

- ▷ Lattice formulation of confluence
- ▷ Lattice formulation of completion

IV. Conclusion and perspectives

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- Rewriting method: present algebraic structures by generators and **oriented** relations.
  - Notion of normal forms.
  - Procedures for computing normal forms.

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- When the presentation of $\mathbf{A}$ is terminating, NF is a generating family of $\mathbf{A}$!

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- When the presentation of $\mathbf{A}$ is confluent and terminating, NF is a linear basis of $\mathbf{A}$!

Gröbner Bases

- Gröbner bases appear in
  - Lie algebras [Shirshov 1962],
  - Commutative algebras [Buchberger 1965],
  - Associative algebras [Bokut 1976, Bergman 1978, Mora 1992],
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- The relations are oriented w.r.t a monomial order $<$:
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- $\mathbf{A} = \mathbb{K}\langle X \mid R\rangle$, where the elements of R are oriented w.r.t a monomial order.
  - R is called a **Gröbner basis** if it induces a confluent presentation.

Objectives

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- Objective: extend the functional approach.
  - We introduce a lattice interpretation of the confluence property.
  - We deduce a lattice interpretation of the completion procedure.
  - The functional approach concerns general linear rewriting systems!

Plan

II. Reduction operators

Definition

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Notations. $\forall v, \ v' \in \mathbb{K}G$

- $\mathrm{lm}\,(v)$: the greatest basis element occurring in the decomposition of v.
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Definition. An endomorphism T of $\mathbb{K}G$ is a **reduction operator** relative to $(G, <)$ if

- T is a projector,
- $\forall g \in G$, we have $T(g) \ \leq \ g$.

Example

- $(G, <) = \{g_1 < g_2 < g_3 < g_4\}$,

$$T_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

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Lattice Structure

Proposition. *The map*

$$\begin{aligned} \ker \colon \mathbf{RO}(G, <) &\longrightarrow \left\{\text{subspaces of } \mathbb{K}G\right\}, \\ T &\longmapsto \ker(T) \end{aligned}$$

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Notation. $\ker^{-1} : \big\{\text{subspaces of } \mathbb{K}G\big\} \longrightarrow \mathbf{RO}\,(G,\ <)$ the inverse of ker.

Lattice structure. $\Big(\mathbf{RO}\,(G,\ <)\,,\ \preceq,\ \wedge,\ \vee\Big)$ is a lattice where

- $T_1 \ \preceq\ T_2$ if $\ker\,(T_2)\ \subseteq\ \ker\,(T_1)$.
- $T_1 \wedge T_2 \ =\ \ker^{-1}(\ker(T_1) + \ker(T_2))$.
- $T_1 \vee T_2 \ =\ \ker^{-1}(\ker(T_1) \cap \ker(T_2))$.

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III. Confluence and completion

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Remark. The obstruction to confluence is $\nexists\, g_3 \longrightarrow g_1$:

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& g_4 \arrow[dl, "T_1"'] \arrow[dr, "T_2"] & \\
{\color{red}g_3} & & g_2 \arrow[dl, "T_1"] \\
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Proposition. *F is confluent if and only if it is so for*

$$\underset{F}{\longrightarrow} = \bigcup_{T \in F} \underset{T}{\longrightarrow}.$$

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- Formally

$$C^P = \begin{pmatrix} 1 & 0 & \mathbf{1} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Lattice description of the completion procedure

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$$C^F(g) = \begin{cases} \wedge F(g), & \text{if } g \in \mathrm{Obs}^F \\ g, & \text{otherwise.} \end{cases}$$

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Theorem. *Letting* $\vee\overline{F} = \ker^{-1}(\mathrm{NF}(F))$, *we have*:

$$C^F = (\wedge F) \vee \left(\vee\overline{F}\right).$$

Plan

IV. Conclusion

Summary and perspectives

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- **Thank you for listening!**