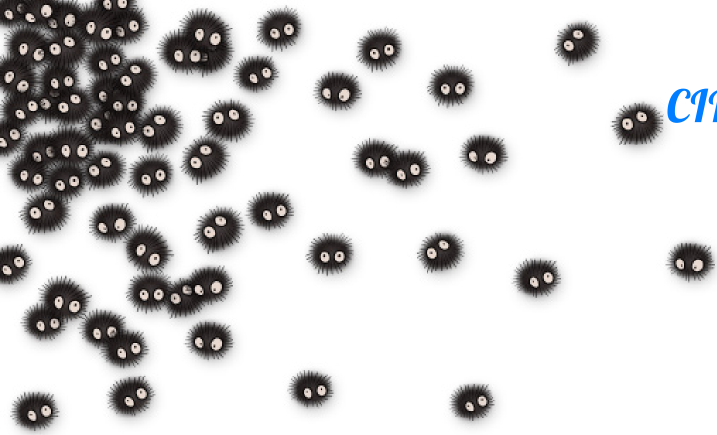
JNCF 2019,

CIRM, Marseille

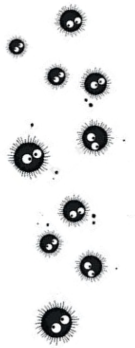


A New Lower Bound on the Hilbert Number for Quartic Systems

F. Bréhard[1,2], N. Brisebarre[1], M. Joldeș[2] and W. Tucker[1,3]

1. LIP, ENS de Lyon 2. LAAS-CNRS, Toulouse 3. CAPA, Uppsala universitet

Outline



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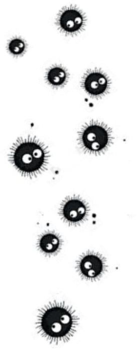

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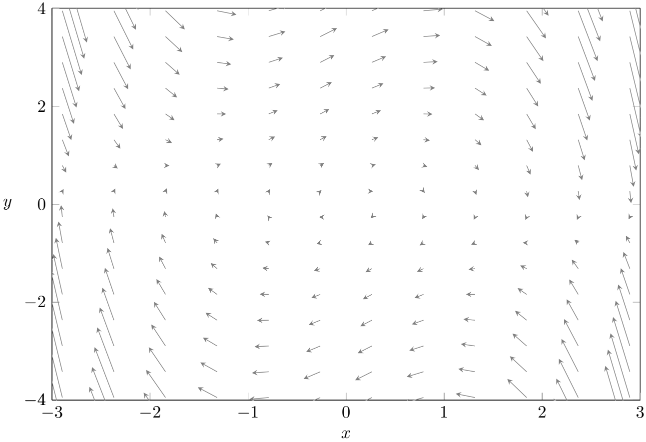
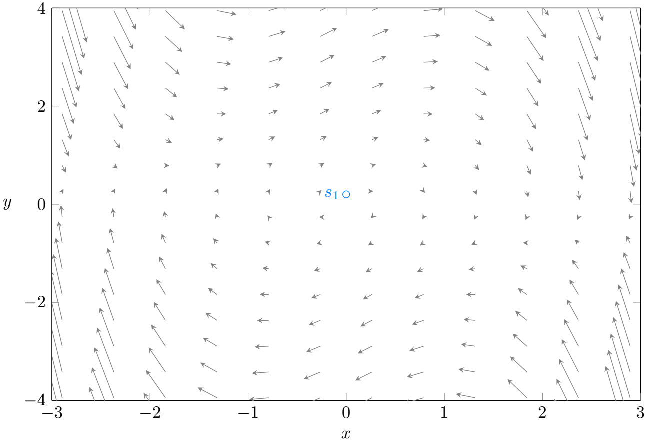


Van der Pol oscillator: $\begin{cases} \dot{x} = y \\ \dot{y} = -x + (1 - x^2)y \end{cases}$



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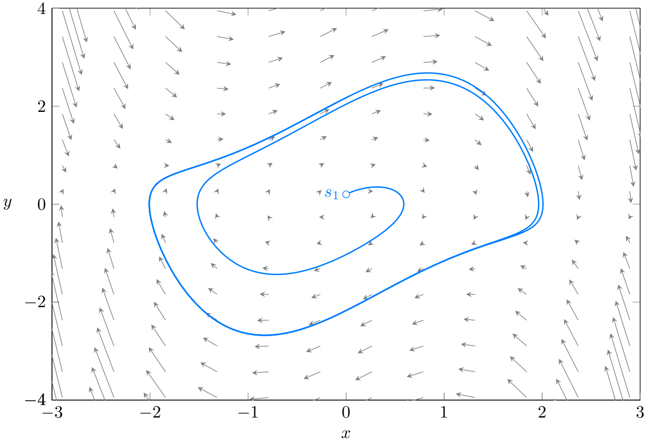

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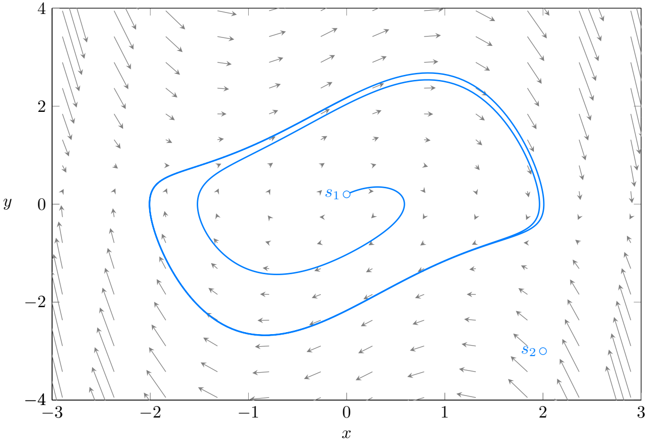

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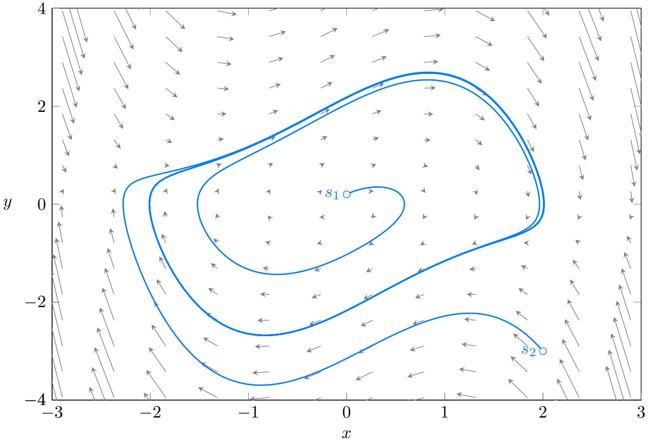

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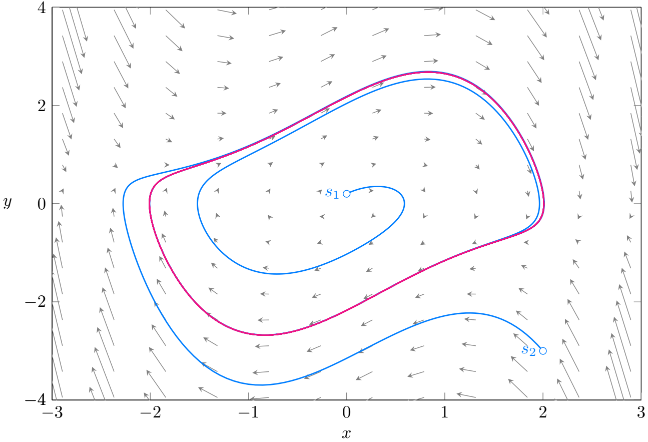

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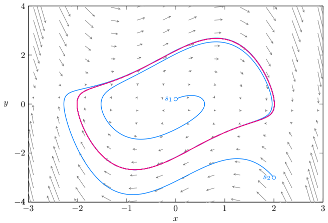

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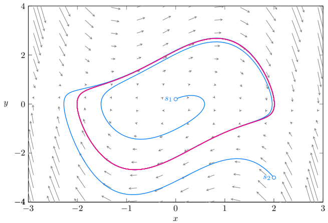

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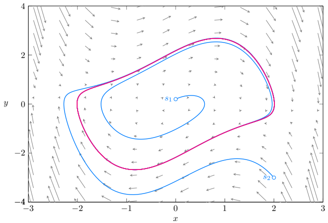

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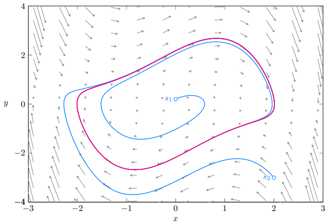
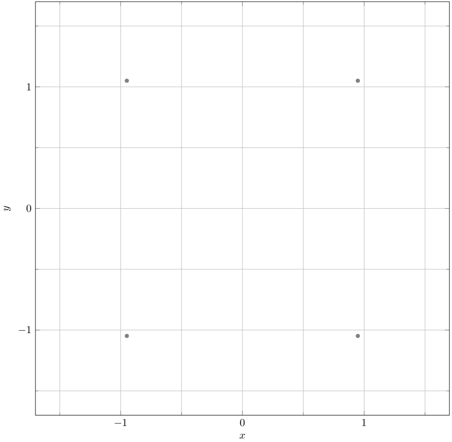
- But even $\mathcal{H}(2) < \infty$ is open!
- Some lower bounds: $\mathcal{H}(2) \geqslant 4$, $\mathcal{H}(3) \geqslant 13$, $\mathcal{H}(4) \geqslant 22$.
- We prove $\boldsymbol{\mathcal{H}(4) \geqslant 24}$.





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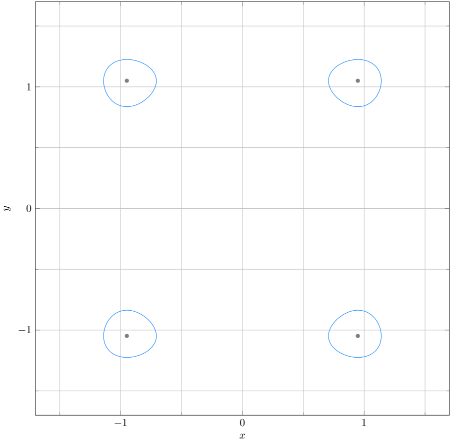
T. Johnson, A quartic system with twenty-six limit cycles, *Experimental Mathematics*, 2011



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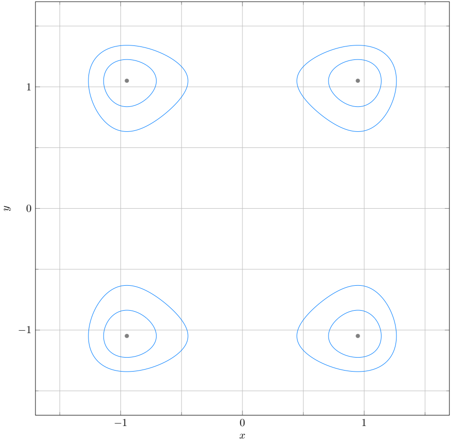

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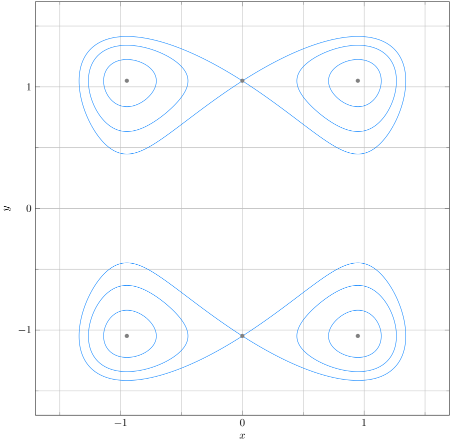

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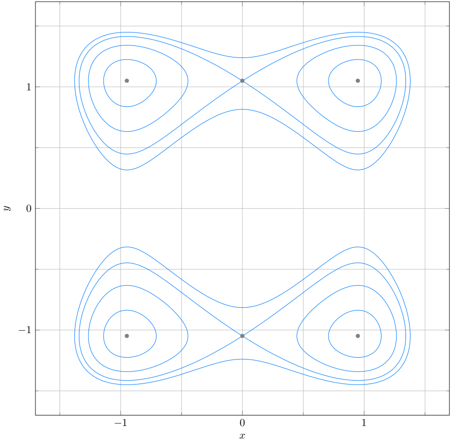

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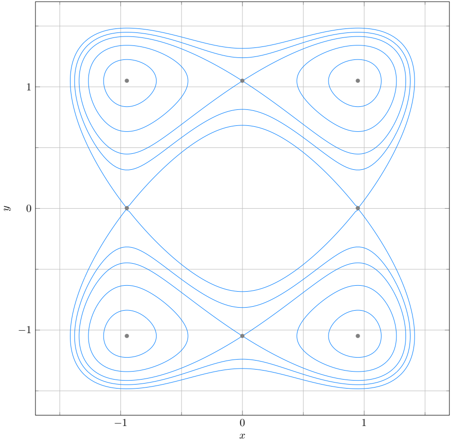

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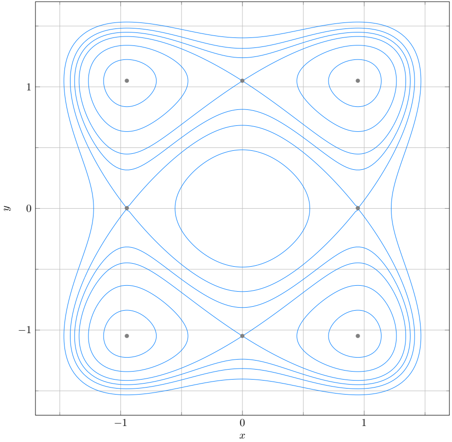

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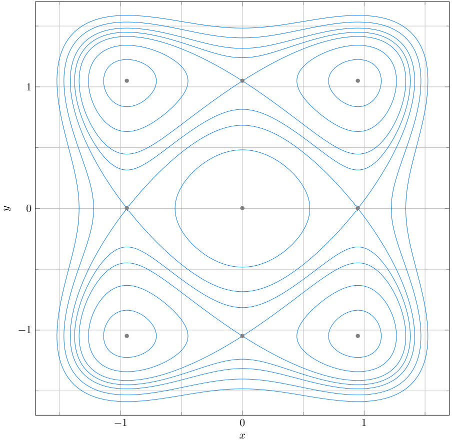

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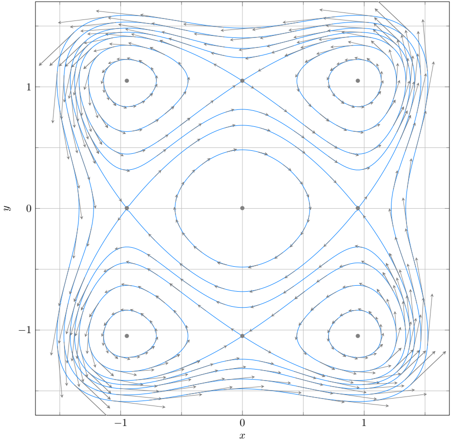

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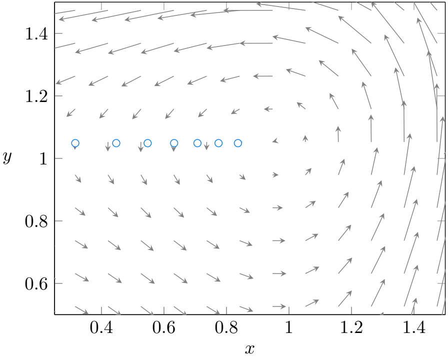

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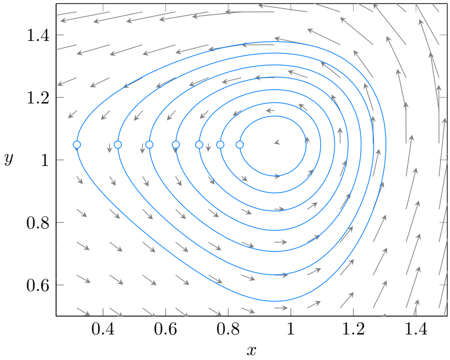

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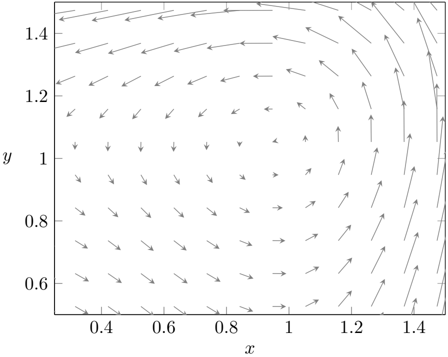

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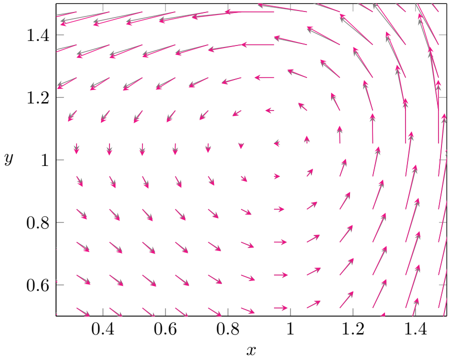

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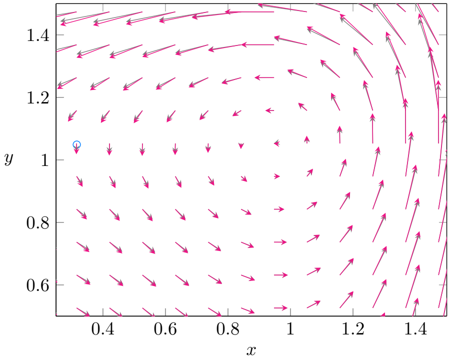

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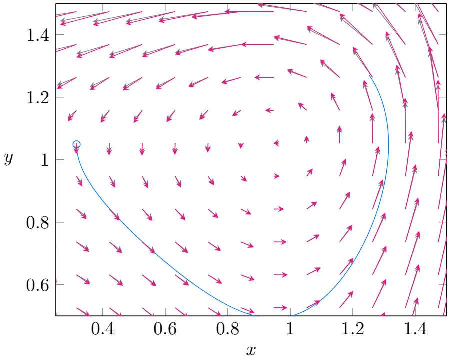

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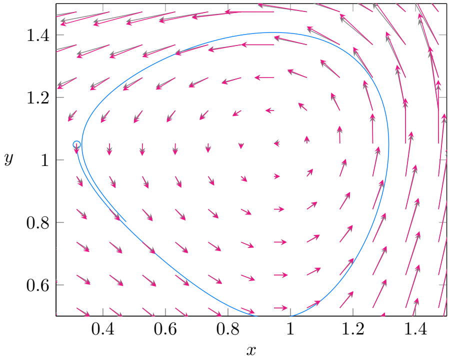

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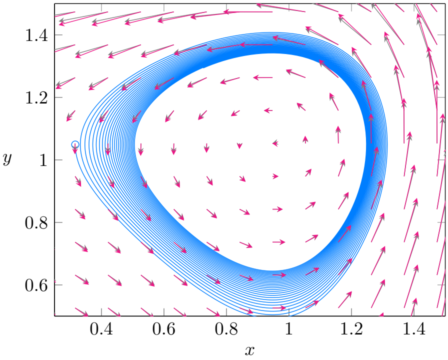

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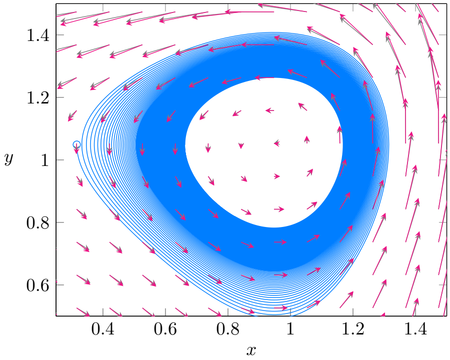

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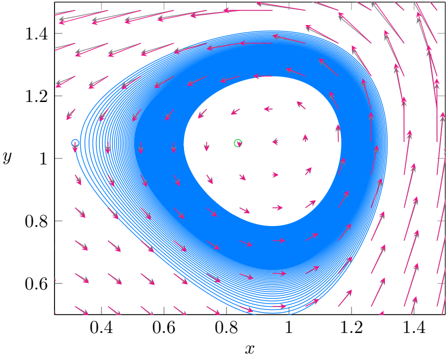

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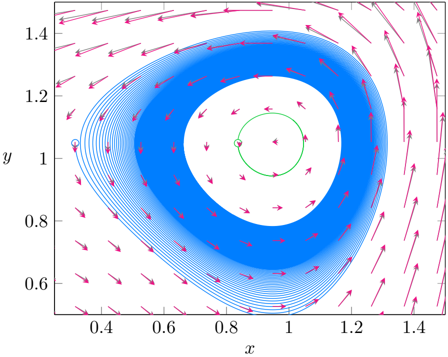

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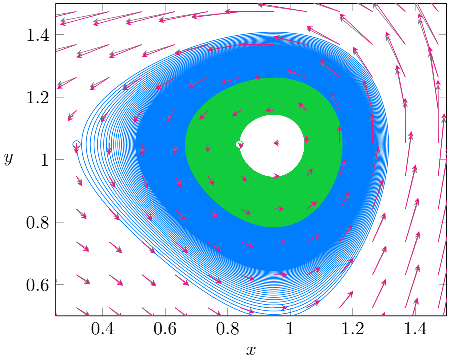

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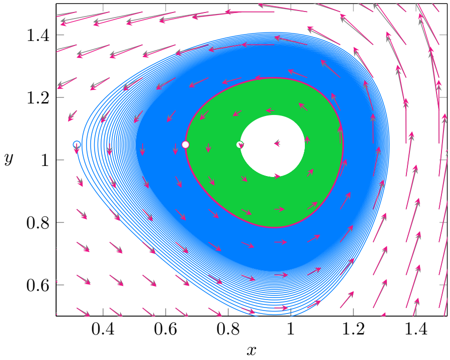

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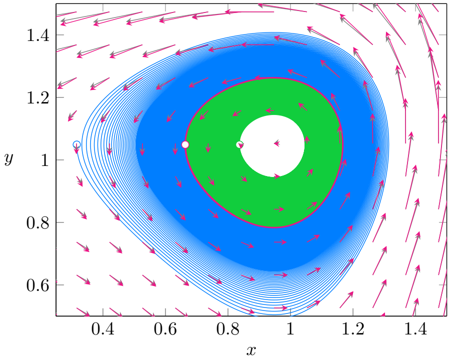

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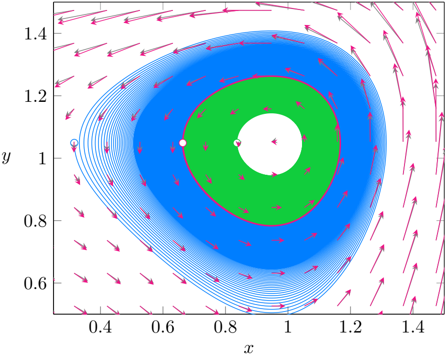
Infinitesimal Hilbert's 16th problem

For a given integer n, what is the maximal number $\mathcal{Z}(n)$ of limit cycles a **perturbed Hamiltonian** vector field of the form:

$$\begin{cases} \dot{x} = -\partial_y H(x,y) + \varepsilon f(x,y) \\ \dot{y} = \partial_x H(x,y) + \varepsilon g(x,y) \end{cases}$$

can have when $\varepsilon \to 0$, with:

- $H(x,y)$ a polynomial potential function of degree $n+1$
- f,g polynomial perturbations of degree n



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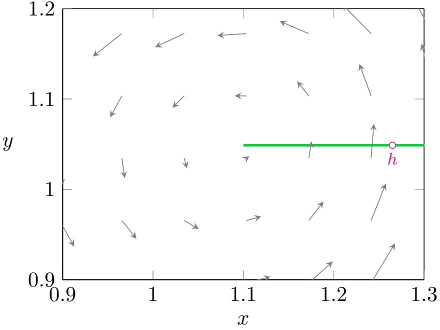
- $\mathcal{Z}(n) < \infty$ for all n
- Pessimistic upper bounds



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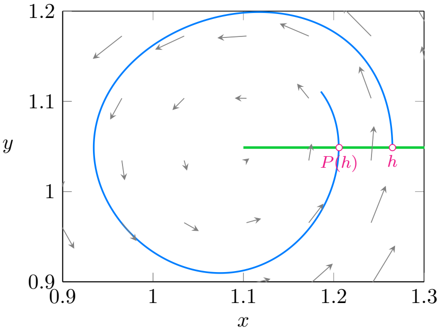

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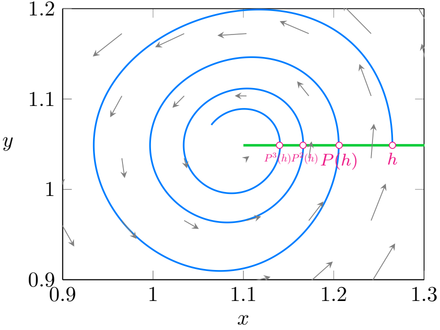

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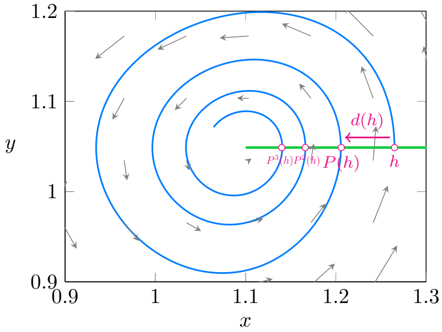

- Poincaré first return map $P(h)$
- Displacement $d(h) = P(h) - h$

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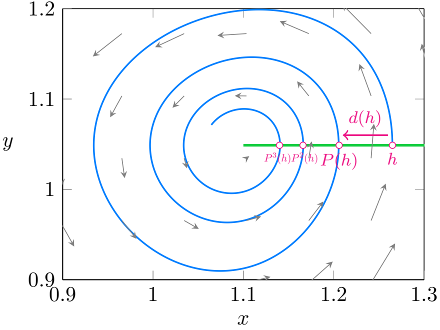

- Poincaré first return map $P(h)$
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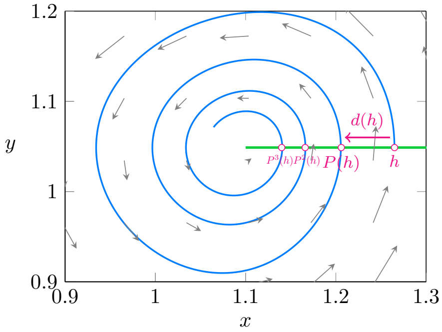

- Poincaré first return map $P(h)$
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- Limit cycle ⇔ isolated zero of d
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$$\oint_{H^{-1}(h)} f(x,y)\mathrm{d}y - g(x,y)\mathrm{d}x$$

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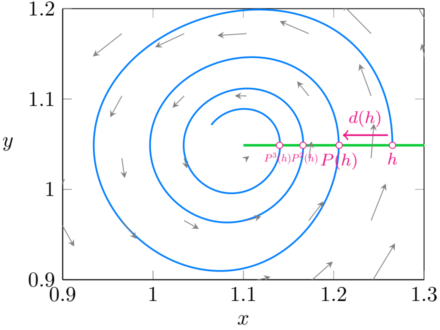

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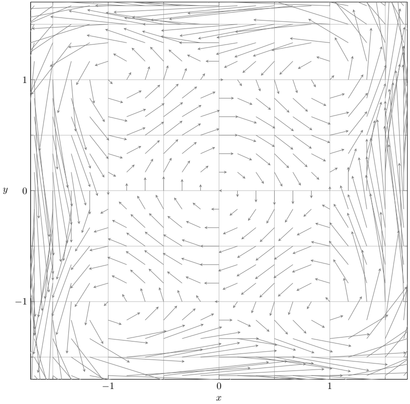
Poincaré-Pontryagin theorem

The Abelian integral $\mathcal{I}(h)$ approximates the displacement function $d(h)$ for small ε:

$$d(h) = \varepsilon(\mathcal{I}(h) + O(\varepsilon)) \qquad \text{when } \varepsilon \to 0$$

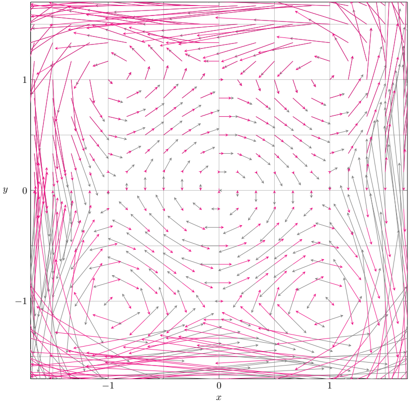
- Hamiltonian system:

$$\begin{cases} \dot{x} = -4y(y^2 - 1.1) \\ \dot{y} = 4x(x^2 - 0.9) \end{cases}$$



- **pseudo-**Hamiltonian system:

$$\begin{cases} \dot{x} = -4\mathbf{y}y(y^2 - 1.1) \\ \dot{y} = 4\mathbf{y}x(x^2 - 0.9) \end{cases}$$



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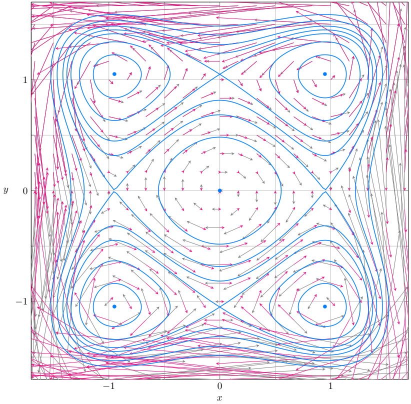

- **pseudo-**Hamiltonian system:

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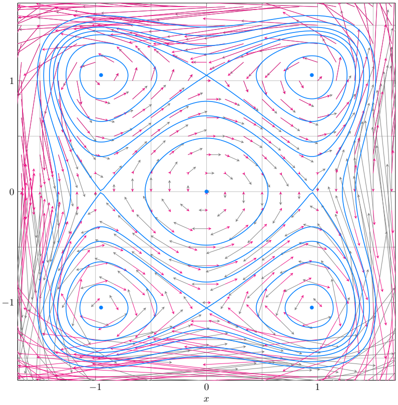
- same geometric orbits after rescaling
- $\simeq$ perturbations without rescaling:

$$\frac{f(x,y)}{\mathbf{y}}, \frac{g(x,y)}{\mathbf{y}} \in \langle x^i y^j, i \geqslant 0, j \geqslant \mathbf{-1}, i+j \leqslant 3 \rangle$$



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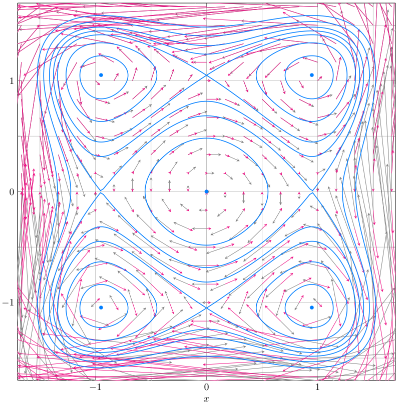

Generalized Poincaré-Pontryagin theorem

The *generalized* Abelian integral:

$$\mathcal{I}(h) = \oint_{H^{-1}(h)} \frac{f(x,y)\mathrm{d}y - g(x,y)\mathrm{d}x}{\mathbf{y}}$$

approximates the displacement function $d(h)$ for small ε:

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- **pseudo-**Hamiltonian system:

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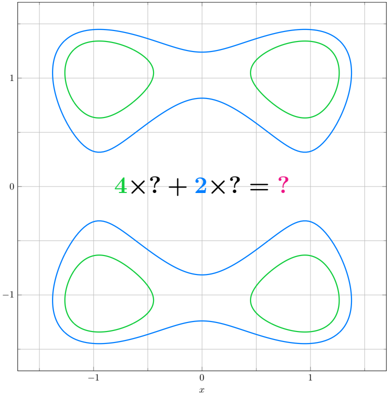

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$\Rightarrow$ The finiteness of $\mathcal{Z}(4)$ does not apply, but we still have some tools of the Hamiltonian case!

Choice of Perturbations

$$f(x,y) = \begin{array}{ccccc} 1 & x & y & x^2 & xy \\ y^2 & x^3 & x^2y & xy^2 & y^3 \\ x^4 & x^3y & x^2y^2 & xy^3 & y^4 \end{array}$$

$$g(x,y) = \begin{array}{ccccc} 1 & x & y & x^2 & xy \\ y^2 & x^3 & x^2y & xy^2 & y^3 \\ x^4 & x^3y & x^2y^2 & xy^3 & y^4 \end{array}$$

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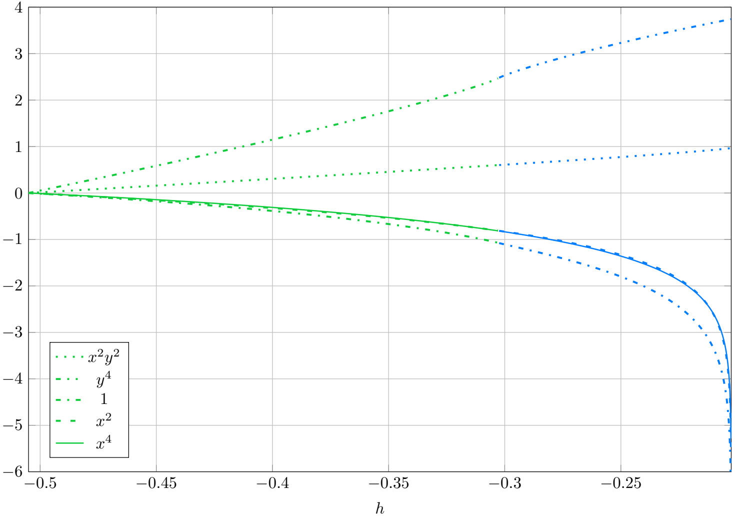
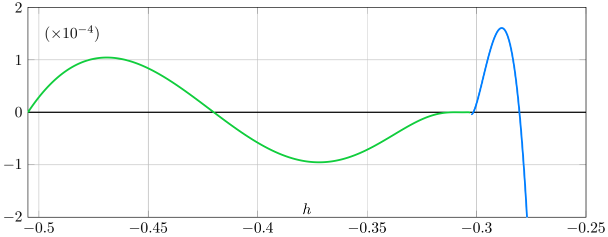
- symmetry requirements
- linear relations from Green's formula: $\partial_x \frac{f(x,y)}{y} \propto \partial_y \frac{g(x,y)}{y}$

$$\mathcal{I}(h) = \oint_{H^{-1}(h)} \frac{\alpha_{00} + \alpha_{20}x^2 + \alpha_{22}x^2y^2 + \alpha_{40}x^4 + \alpha_{04}y^4}{y} \mathrm{d}x$$

- Find coefficients of $\mathcal{I}(h) = \alpha_{00}\mathcal{I}_{00}(h) + \alpha_{20}\mathcal{I}_{20}(h) + \alpha_{22}\mathcal{I}_{22}(h) + \alpha_{40}\mathcal{I}_{40}(h) + \alpha_{04}\mathcal{I}_{04}(h)$.



α_{00}	$=$	-0.78622148667854837664
α_{20}	$=$	0.87723523612653436051
α_{22}	$=$	1
α_{40}	$=$	0.23742713894293038223
α_{04}	$=$	-0.21823846173078863753



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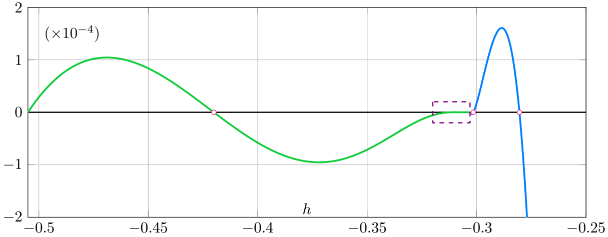

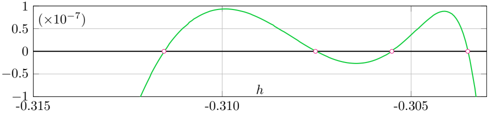

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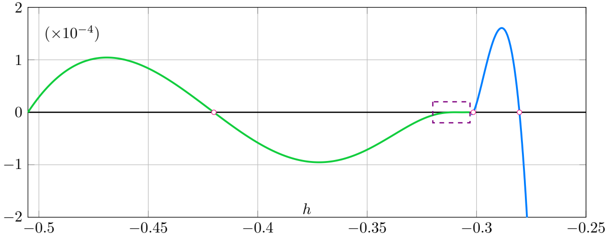
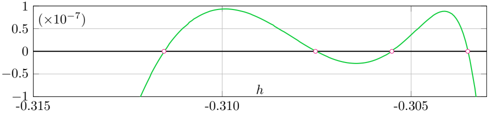
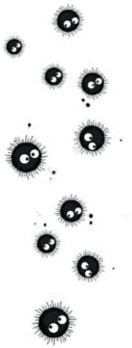
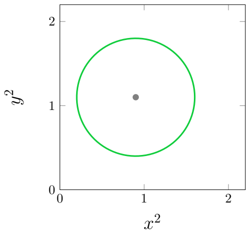
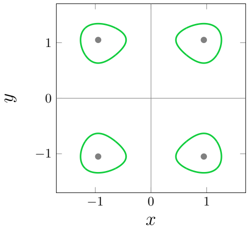




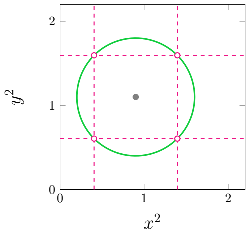
$$4 \times 5 + 2 \times 2 = 24$$

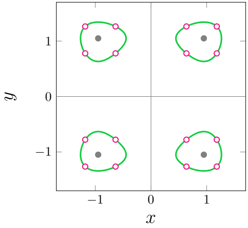






$0 < r \ (= \sqrt{h}) < 0.9$





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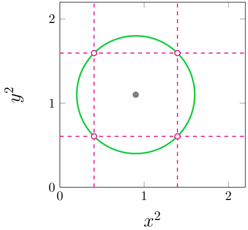

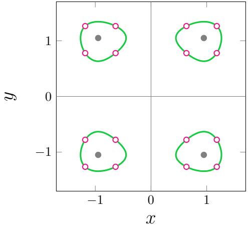

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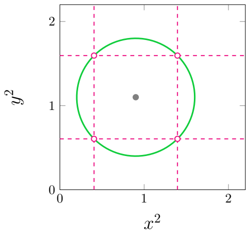
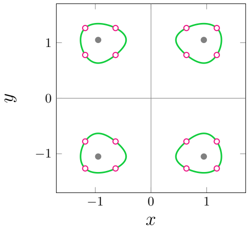
Computing Abelian Integrals





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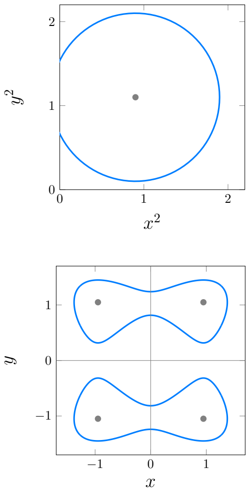
$$\mathcal{I}(h) = \oint_{H^{-1}(h)} \frac{g(x,y)}{y} \mathrm{d}x = \int_{x_{\text{min}}}^{x_{\text{max}}} \left(\frac{g(x, y_{\text{up}}(x))}{y_{\text{up}}(x)} - \frac{g(x, y_{\text{down}}(x))}{y_{\text{down}}(x)} \right) \cdots$$

$$+ \int_{y_{\text{min}}}^{y_{\text{max}}} \left(\frac{g(x_{\text{left}}(y), y)}{x_{\text{left}}(y)} + \frac{g(x_{\text{right}}(y), y)}{x_{\text{right}}(y)} \right) \frac{y^2 - 1.1}{\sqrt{r^2 - (y^2 - 1.1)^2}} \mathrm{d}y.$$



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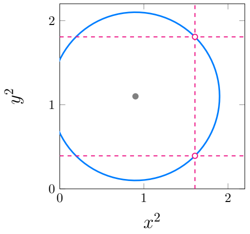

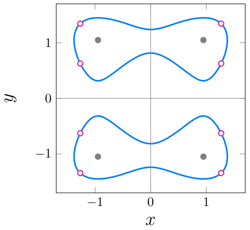

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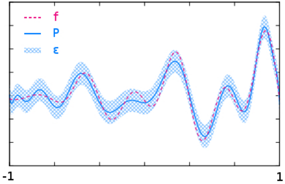
$$+\, 2 \int_{y_{\min}}^{y_{\max}} \frac{g(x_{\mathrm{right}}(y), y)(y^2 - 1.1)}{x_{\mathrm{right}}(y)\sqrt{r^2 - (y^2 - 1.1)^2}}\,\mathrm{d}y.$$

Definition

A pair $(P, \varepsilon) \in \mathbb{R}[X] \times \mathbb{R}_+$ is a rigorous polynomial approximation (RPA) of f for a given norm $\|\cdot\|$ if $\|f - P\| \leqslant \varepsilon$.



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Example: sup-norm over $[-1,1]$:

$$f \in (P, \varepsilon) \Leftrightarrow |f(t) - P(t)| \leqslant \varepsilon \quad \forall t \in [-1,1]$$

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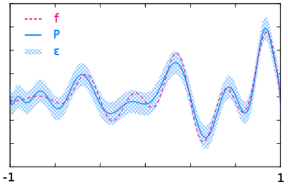
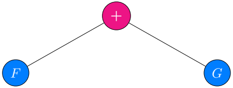


Example:

$$r(t) = f(t) + g(t)$$

Some elementary operations:

- $(P, \varepsilon) + (Q, \eta) := (P + Q, \varepsilon + \eta)$,



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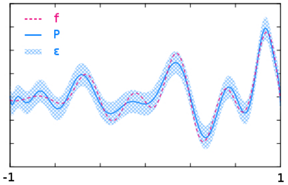

Example:
$$r(t) = f(t) + g(t) - h(t)$$

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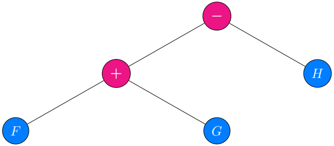

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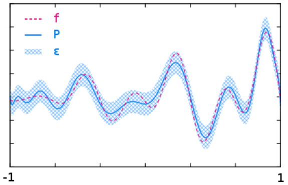

Example: sup-norm over $[-1, 1]$:

$$f \in (P, \varepsilon) \Leftrightarrow |f(t) - P(t)| \leqslant \varepsilon \quad \forall t \in [-1, 1]$$

Example:

$$r(t) = k(t)(f(t) + g(t) - h(t))$$

Some elementary operations:

- $(P, \varepsilon) + (Q, \eta) := (P + Q, \varepsilon + \eta)$,
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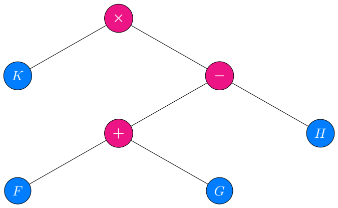
- $(P, \varepsilon) \cdot (Q, \eta) := (PQ, \|Q\|\eta + \|P\|\varepsilon + \eta\varepsilon)$



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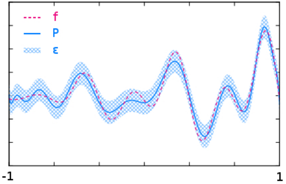

Example: sup-norm over $[-1,1]$:

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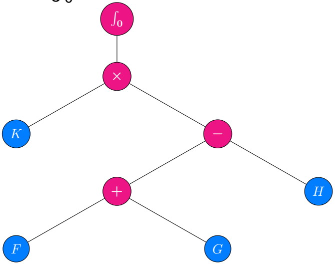
$$r(t) = \int_0^t k(s)(f(s) + g(s) - h(s))\mathrm{d}s$$



Some elementary operations:

- $(P,\varepsilon) + (Q,\eta) := (P+Q, \varepsilon+\eta)$,
- $(P,\varepsilon) - (Q,\eta) := (P-Q, \varepsilon+\eta)$,
- $(P,\varepsilon) \cdot (Q,\eta) := (PQ, \|Q\|\eta + \|P\|\varepsilon + \eta\varepsilon)$
- $\int_0 (P,\varepsilon) := (\int_0^t P(s)\mathrm{d}s, \varepsilon)$

Rigorous Polynomial Approximations

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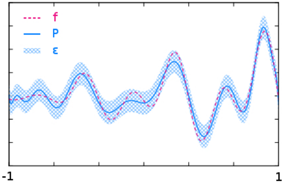

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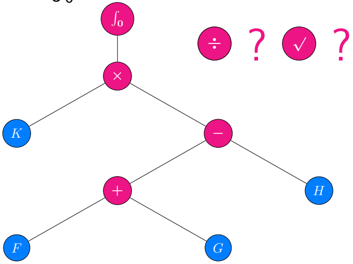

General scheme

- Fixed-point equation $\mathbf{T} \cdot \varphi = \varphi$ with $\mathbf{T}$ contracting,
- Approximation φ° to exact solution $\varphi^\star$,
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If (X, d) is complete and $\mathbf{T}$ **contracting** of ratio $\lambda < 1$,

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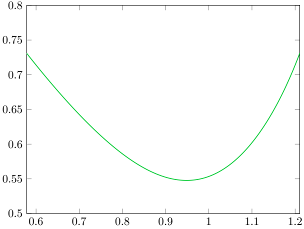
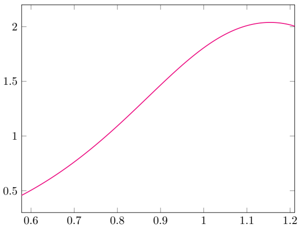
- Applications to numerous function space problems.

- Approximation $\varphi^\circ(x)$ of $\varphi^\star = x^2/y_{\text{down}}(x)$ using Chebyshev interpolation:





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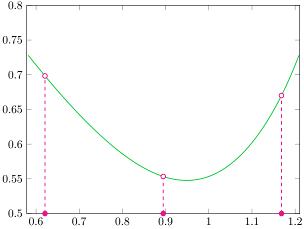

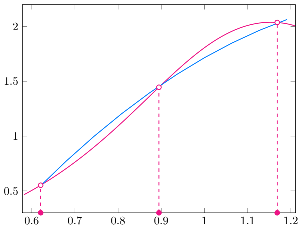

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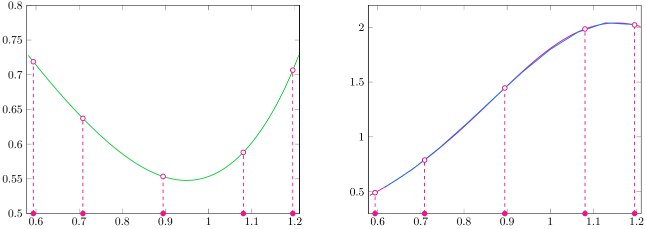

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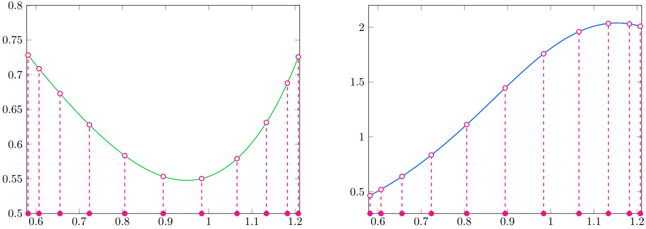

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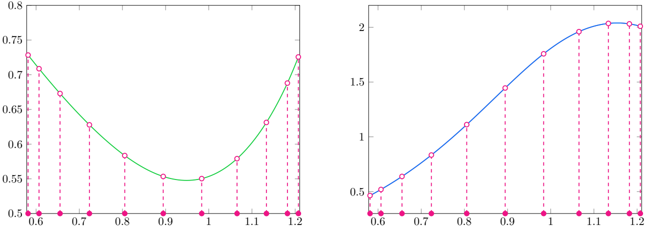
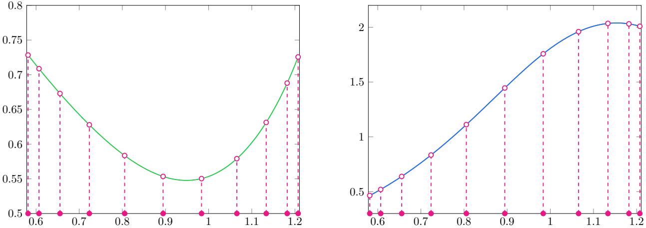


- Newton-like operator $\mathbf{T}$ with unique fixed point $\varphi^{\star} = \dfrac{x^2}{y_{\mathrm{down}}(x)}$:

$$\mathbf{T}\cdot\varphi = \varphi - \psi(y_{\mathrm{down}}\varphi - x^2) \qquad \psi(x) \approx 1/y_{\mathrm{down}}(x)$$

Division of RPAs

- Approximation $\varphi^\circ(x)$ of $\varphi^\star = x^2/y_{\text{down}}(x)$ using Chebyshev interpolation:



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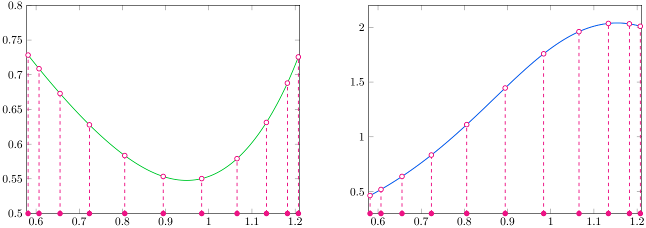

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$$\|\varphi^\circ - \mathbf{T}\cdot\varphi^\circ\| = \|\psi(y_{\text{down}}\varphi^\circ - x^2)\| \leqslant \eta$$

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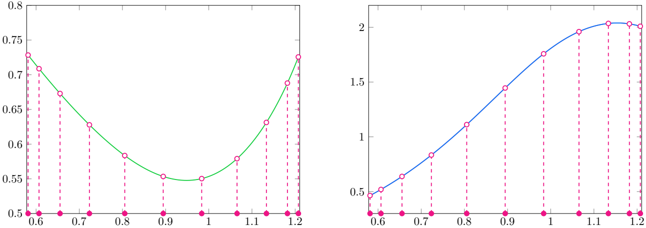

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Square Root of a RPA

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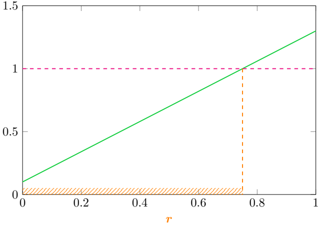
$$\lambda = \sup_{\|\varphi - \varphi^\circ\| \leqslant r} \|\mathcal{D}\mathbf{T}(\varphi)\| \leqslant \|1 - \psi\varphi^\circ\| + \|\psi\| r$$



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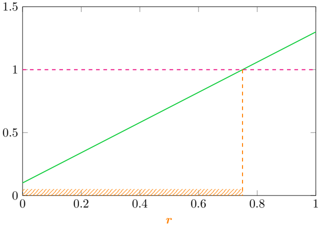

- Stable neighborhood for φ°:

$$\|\varphi^\circ - \mathbf{T} \cdot \varphi^\circ\| + \boldsymbol{\lambda r} \leqslant \boldsymbol{r}$$

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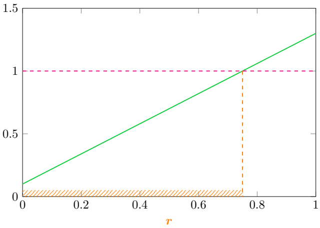

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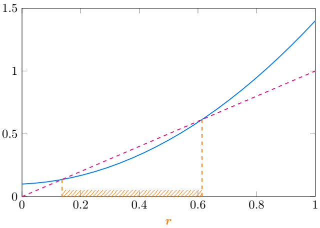
$$\|\psi({\varphi^\circ}^2 - f)/2\| + \boldsymbol{r}(\|1 - \psi\varphi^\circ\| + \|\psi\|\boldsymbol{r}) \leqslant \boldsymbol{r}$$



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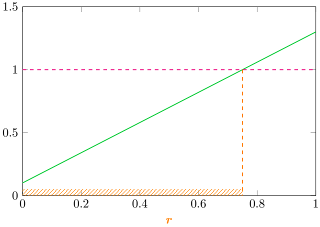

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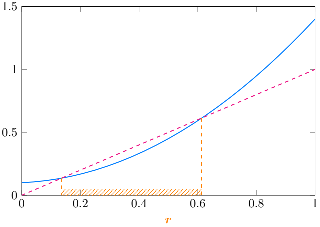
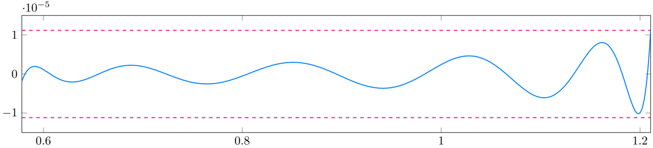


- Apply the Banach fixed-point theorem!

- $\sqrt{0.8 - (x^2 - 0.9)^2}$:



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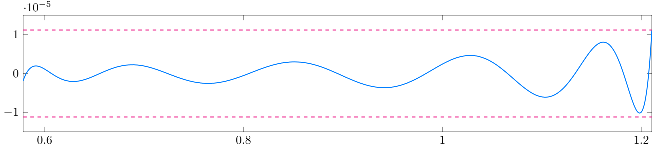
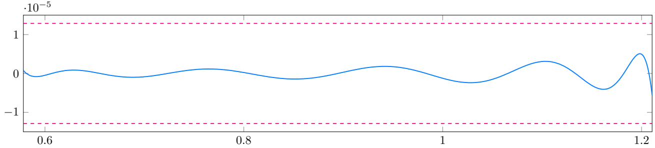


- $y_{\mathrm{down}}(x) = \sqrt{1.1 - \sqrt{0.8 - (x^2 - 0.9)^2}}$:



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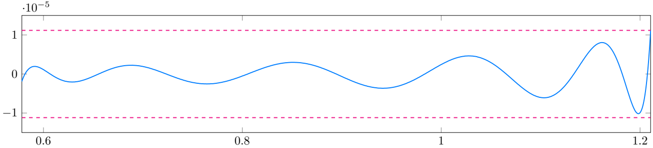

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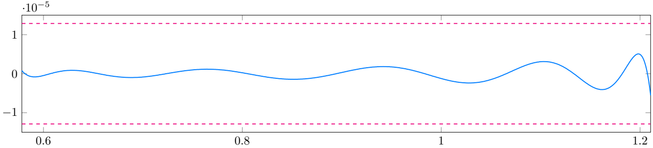
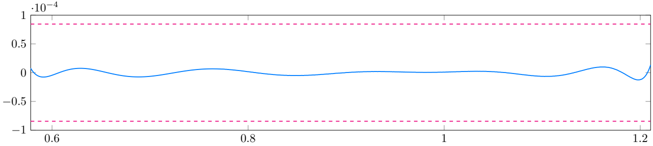
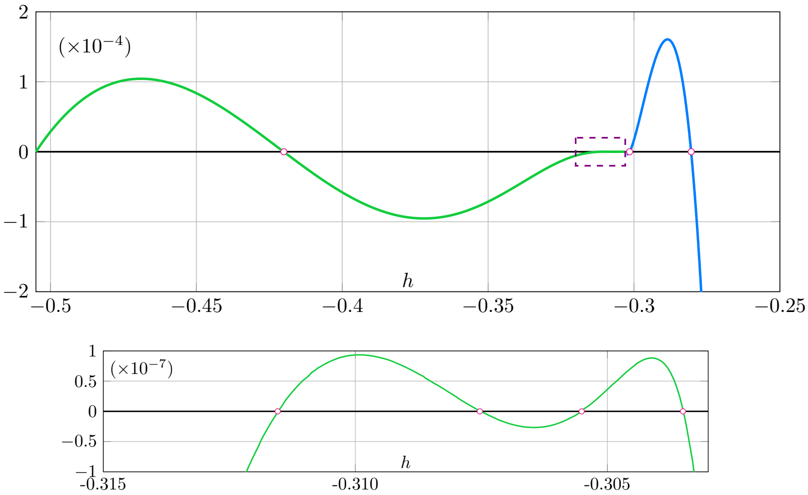
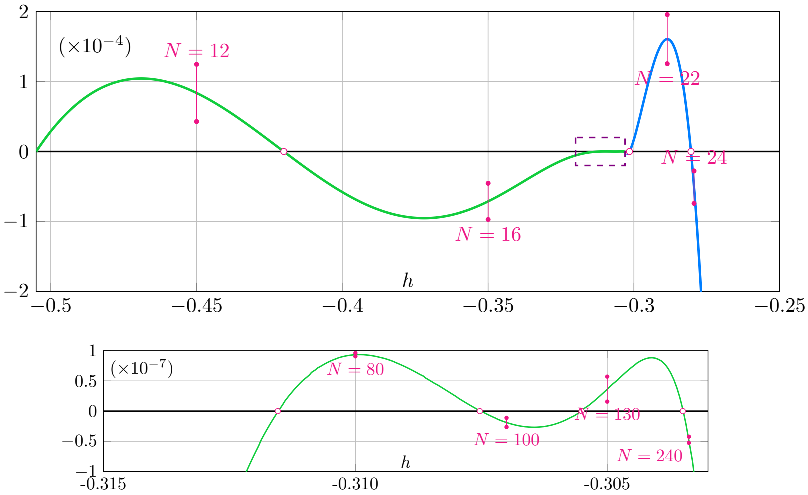


- $x^2/y_{\text{down}}(x) = x^2/y_{\text{down}}(x) = \sqrt{1.1 - \sqrt{0.8 - (x^2 - 0.9)^2}}$:



Validation of Our Result

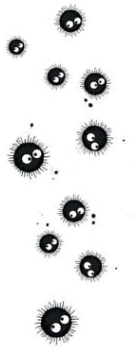




2
1
0
−1
−2
(×10^−4)
N = 12
N = 22
N = 24
N = 16
h
−0.5
−0.45
−0.4
−0.35
−0.3
−0.25
1
0.5
0
−0.5
−1
(×10^−7)
N = 80
N = 100
N = 130
N = 240
h
-0.315
-0.310
-0.305

$$4 \times 5 + 2 \times 2 = 24$$

Outline



- $(f_0, f_1, \ldots, f_n)$ **extended Chebyshev system** if all combination $\alpha_0 f_0 + \cdots + \alpha_i f_i$ has at most i zeros, for $0 \leqslant i \leqslant n$.

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- $\Leftrightarrow$ $W_0(x), \ldots, W_n(x) \neq 0$ for all x:

$$W_i(x) = \begin{vmatrix} f_0(x) & f_1(x) & \ldots & f_i(x) \\ f_0'(x) & f_1'(x) & \ldots & f_i'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(i)}(x) & f_1^{(i)}(x) & \ldots & f_i^{(i)}(x) \end{vmatrix}$$

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- In our case, W_0, W_1, W_2, W_3 never vanish on small ovals and W_4 does once. $\Rightarrow$ at most 5 zeros on small ovals.

- $(f_0, f_1, \dots, f_n)$ **extended Chebyshev system** if all combination $\alpha_0 f_0 + \cdots + \alpha_i f_i$ has at most i zeros, for $0 \leqslant i \leqslant n$.

- $\Leftrightarrow$ $W_0(x), \dots, W_n(x) \neq 0$ for all x:

$$W_i(x) = \begin{vmatrix} f_0(x) & f_1(x) & \dots & f_i(x) \\ f_0'(x) & f_1'(x) & \dots & f_i'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(i)}(x) & f_1^{(i)}(x) & \dots & f_i^{(i)}(x) \end{vmatrix}$$

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- but no **rigorous** proof!

$$\oint_{H=h} \frac{f(x,y)\mathrm{d}y - g(x,y)\mathrm{d}x}{y} = \iint_{H \leqslant h} \left(\partial_x \frac{f(x,y)}{y} + \partial_y \frac{g(x,y)}{y} \right) \mathrm{d}x\mathrm{d}y$$

$$\oint_{H=h} \frac{f(x,y)\mathrm{d}y - g(x,y)\mathrm{d}x}{y} = \iint_{\mathbb{R}^2} \left(\partial_x \frac{f(x,y)}{y} + \partial_y \frac{g(x,y)}{y} \right) \mathbf{1}_{h-H}(x,y,h)\mathrm{d}x\mathrm{d}y$$

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- Hence, so does $\left(\partial_x \frac{f(x,y)}{y} + \partial_y \frac{g(x,y)}{y}\right) \mathbf{1}_{h-H}(x,y,h)$.
- By creative telescoping, one finds a differential equation for $\mathcal{I}(h)$.

```
DiffEq[4, 0]
```

$$\left(-128 h^5 X0^2 + 336 h^4 X0^4 - 288 h^3 X0^6 + 80 h^2 X0^8 - 64 h^5 Y0^2 + 480 h^4 X0^2 Y0^2 - 800 h^3 X0^4 Y0^2 + 464 h^2 X0^6 Y0^2 - 80 h X0^8 Y0^2 + 144 h^4 Y0^4 - 544 h^3 X0^2 Y0^4 + 576 h^2 X0^4 Y0^4 - 176 h X0^6 Y0^4 - 96 h^3 Y0^6 + 208 h^2 X0^2 Y0^6 - 112 h X0^4 Y0^6 + 16 h^2 Y0^8 - 16 h X0^2 Y0^8\right) D_h^4 +$$
$$\left(-640 h^4 X0^2 + 1632 h^3 X0^4 - 1392 h^2 X0^6 + 400 h X0^8 - 320 h^4 Y0^2 + 1920 h^3 X0^2 Y0^2 - 2848 h^2 X0^4 Y0^2 + 1408 h X0^6 Y0^2 - 160 X0^8 Y0^2 + 480 h^3 Y0^4 - 1520 h^2 X0^2 Y0^4 + 1392 h X0^4 Y0^4 - 352 X0^6 Y0^4 - 192 h^2 Y0^6 + 416 h X0^2 Y0^6 - 224 X0^4 Y0^6 + 32 h Y0^8 - 32 X0^2 Y0^8\right) D_h^3 +$$
$$\left(-480 h^3 X0^2 + 1188 h^2 X0^4 - 1020 h X0^6 + 315 X0^8 - 240 h^3 Y0^2 + 1080 h^2 X0^2 Y0^2 - 1512 h X0^4 Y0^2 + 660 X0^6 Y0^2 + 180 h^2 Y0^4 - 540 h X0^2 Y0^4 + 378 X0^4 Y0^4 - 48 h Y0^6 + 36 X0^2 Y0^6 + 3 Y0^8\right) D_h^2$$

$(X_0 = 0.9, Y_0 = 1.1)$

- Rigorous Polynomial Approximations at ordinary points[1]

[1] F. Bréhard, N. Brisebarre, M. Joldes. *Validated and Numerically Efficient Chebyshev Spectral Methods for Linear Ordinary Differential Equations*. ACM Transactions on Mathematical Software (TOMS), 2018.

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```

- Analytic at $h = 0$
  $\Rightarrow$ Initial conditions with **Laplace transform**
- log singularities at $h = X_0^2 \ldots$
  $\Rightarrow$ Other rigorous approximation tools.

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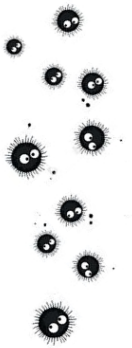
Outline



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- **Future work:** Generalization to other systems?