From moments to sparse representations, an algebraic, geometric and algorithmic viewpoint

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Chapter 1

Sparse representations from moments

1.1 Prony's method in one variables

One of the first work in this area is probably due to Gaspard-Clair-François-Marie Riche de Prony, mathematician and engineer of the École Nationale des Ponts et Chaussées. He was working on Hydraulics. To analyze the expansion of various gases, he proposed in [dP95] a method to fit a sum of exponentials at equally spaced data points in order to extend the model at intermediate points. More precisely, he was studying the following problem:

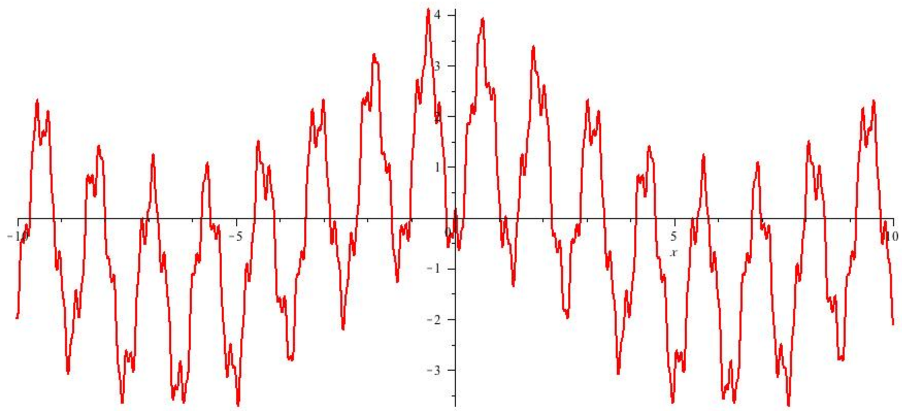
Given a function $h \in C^{\infty}(\mathbb{R})$ of the form

$$x \in \mathbb{R} \mapsto h(x) = \sum_{i=1}^{r} \omega_i \, e^{f_i x} \in \mathbb{C} \tag{1.1}$$

where $f_1, \ldots, f_r \in \mathbb{C}$ are pairwise distinct, $\omega_i \in \mathbb{C} \setminus \{0\}$, the problem consists in recovering

- the distinct *frequencies* $f_1, \ldots, f_r \in \mathbb{C}$,
- the *coefficients* $\omega_i \in \mathbb{C} \setminus \{0\}$,

Here is an example of such a signal, which is the superposition of several "oscillations" with different frequencies.



The approach proposed by G. de Prony can be reformulated into a truncated series reconstruction problem. By choosing an arithmetic progression of points in $\mathbb{R}$, for instance the integers $\mathbb{N}$, we can associate to h, the generating series:

$$\sigma_h(y) = \sum_{a\in\mathbb{N}} h(a)\frac{y^a}{a!} \in \mathbb{C}[[y]],$$

where $\mathbb{C}[[y]]$ is the ring of formal power series in the variable y. If h is of the form (1.1), then

$$\sigma_h(y) = \sum_{i=1}^{r}\sum_{a\in\mathbb{N}} \omega_i \xi_i^a \frac{y^a}{a!} = \sum_{i=1}^{r} \omega_i e^{\xi_i y} \tag{1.2}$$

where $\xi_i = e^{f_i}$. Prony's method consists in reconstructing the decomposition (1.2) from a small number of coefficients $h(a)$ for $a = 0,\ldots,2r-1$. It performs as follows:

- From the values $h(a)$ for $a \in [0,\ldots,2r-1]$, compute the polynomial

  $$p(x) = \prod_{i=1}^{r}(x-\xi_i) = x^r - \sum_{j=0}^{r-1} p_j x^j,$$

  which roots are $\xi_i = e^{f_i}$, $i = 1,\ldots,r$ as follows. Since it satisfies the recurrence relations

  $$\forall j \in [0,\ldots,r-1], \qquad \sum_{i=0}^{r-1} \sigma_{j+i}p_i - \sigma_{j+r} = -\sum_{i=1}^{r} w_i \xi_i^j p(\xi_i) = 0,$$

  it is the unique solution of the system:

  $$\begin{pmatrix} \sigma_0 & \sigma_1 & \cdots & & \sigma_{r-1} \\ \sigma_1 & & & \iddots & \\ \vdots & & \iddots & & \vdots \\ & \iddots & & & \\ \sigma_{r-1} & & \cdots & & \sigma_{2r-2} \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ \vdots \\ p_{r-1} \end{pmatrix} = \begin{pmatrix} \sigma_r \\ \sigma_{r+1} \\ \vdots \\ \vdots \\ \sigma_{2r-1} \end{pmatrix}. \tag{1.3}$$

- Compute the roots $\xi_1, \ldots, \xi_r$ of the polynomial $p(x)$.
- To determine the weight coefficients $w_1, \ldots, w_r$, solve the following linear (Vandermonde) system:

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ \xi_1 & \xi_2 & \ldots & \xi_r \\ \vdots & \vdots & & \vdots \\ \xi_1^{r-1} & \xi_2^{r-1} & \ldots & \xi_r^{r-1} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_r \end{pmatrix} = \begin{pmatrix} h_0 \\ h_1 \\ \vdots \\ h_{r-1} \end{pmatrix}.$$

This approach can be improved by computing the roots $\xi_1, \ldots, \xi_r$, directly as the generalized eigenvalues of a pencil of Hankel matrices. Namely, Equation (1.3) implies that

$$\overbrace{\begin{pmatrix} \sigma_0 & \sigma_1 & \ldots & & \sigma_{r-1} \\ \sigma_1 & & & \iddots & \\ \vdots & & \iddots & & \vdots \\ & \iddots & & & \\ \sigma_{r-1} & & \ldots & & \sigma_{2r-2} \end{pmatrix}}^{H_0} \overbrace{\begin{pmatrix} 0 & & & & p_0 \\ 1 & \ddots & & & p_1 \\ & \ddots & \ddots & & \vdots \\ & & \ddots & 0 & \vdots \\ & & & 1 & p_{r-1} \end{pmatrix}}^{M_p} = \overbrace{\begin{pmatrix} \sigma_1 & \sigma_2 & \ldots & & \sigma_r \\ \sigma_2 & & & \iddots & \\ \vdots & & \iddots & & \vdots \\ & \iddots & & & \\ \sigma_r & & \ldots & & \sigma_{2r-1} \end{pmatrix}}^{H_1}, \tag{1.4}$$

so that the generalized eigenvalues of the pencil (H_1, H_0) are the eigenvalues of the companion matrix M_p of $p(x)$, that is, its the roots $\xi_1, \ldots, \xi_r$. This variant of Prony's method is also called the *pencil method* in the literature.

For numerical improvement purposes, one can also chose an arithmetic progression $\frac{a}{T}$ and $a \in [0, \ldots, 2r-1]$, with $T \in \mathbb{R}^+$ of the same order of magnitude as the frequencies f_i. The roots of the polynomial p are then $\xi_i = e^{\frac{f_i}{T}}$.

1.2 Symmetric tensor decomposition

Symmetric tensors of order d of a vector space V of dimension $n+1$ over a field $\mathbb{K}$ are the elements of the symmetric product $S^{(d)}(V)$. Once we have chosen a basis of V, these elements can be identified with homogeneous polynomials of degree d in $n+1$ variables $x_0, x_1, \ldots, x_n$. Let $S = \mathbb{K}[x_0, x_1, \ldots, x_n] = \mathbb{K}[\overline{\boldsymbol{x}}]$ be the ring of polynomials in these variables. The set of symmetric tensors of degree $d \in \mathbb{N}$ is the vector space S_d of homogenous polynomials of degree d. An element $\psi \in S_d$ is of the form $\psi = \sum_{|\alpha|=d} \sigma_\alpha \binom{d}{\alpha} \overline{\boldsymbol{x}}^\alpha$ where $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1}$, $\overline{\boldsymbol{x}}^\alpha = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$, $\binom{d}{\alpha} = \frac{d!}{\alpha_0! \cdots \alpha_n!}$ for $|\alpha| = \alpha_0 + \cdots + \alpha_n = d$.

The tensor decomposition problem is the following:

Problem 1.2.1 (Symmetric tensor decomposition)

Given a homogeneous polynomial

$$\psi(\overline{\boldsymbol{x}}) = \sum_{|\alpha|=d} \sigma_\alpha \binom{d}{\alpha} \overline{\boldsymbol{x}}^\alpha$$

of degree d in the variables $\overline{\boldsymbol{x}} = (x_0, x_1, \ldots, x_n)$, find a decomposition of ψ of the form

$$\psi(\overline{\boldsymbol{x}}) = \sum_{i=1}^{r} \omega_i (\xi_{i,0} x_0 + \xi_{i,1} x_1 + \cdots + \xi_{i,n} x_n)^d$$

where $\xi_i = (\xi_{i,0}, \xi_{i,1}, \ldots, \xi_{i,n})$, $i = 1, \ldots, r$ span distinct lines in $\overline{\mathbb{K}}^{n+1}$ and $\omega_i \in \overline{\mathbb{K}}$.

The minimal r in such a decomposition is called the *rank* of ψ.

In the decomposition, the vectors ξ_i span distinct lines in $\overline{\mathbb{K}}^{n+1}$, which means that they define distinct points in $\mathbb{P}(\overline{\mathbb{K}}^{n+1})$. In particular, they are non-zero vectors.

Example 1.2.2 *Consider a quadratic form $q(\overline{\boldsymbol{x}}) = \sum_{0 \leqslant i,j \leqslant n} q_{i,j} x_i x_j \in \mathbb{R}[\overline{\boldsymbol{x}}]_2$ with $q_{i,j} = q_{j,i}$. Using a classical reduction of quadratic forms into weighted sums of squares (e.g. $ax^2 + 2bxy + cy^2 = a(x + \frac{b}{a}y)^2 + (c - \frac{b^2}{a})y^2$ if $a \neq 0$), q can be written as*

$$q(\overline{\boldsymbol{x}}) = \sum_{i=1}^{r} \omega_i (\xi_{i,0} x_0 + \cdots + \xi_{i,n} x_n)^2$$

with $\omega_i \in \mathbb{K}$ and $\xi_i = (\xi_{i,0}, \ldots, \xi_{i,n})$ are distinct in $\mathbb{P}^n$. The minimal number of terms in this decomposition is known to be the rank of q, or equivalently, the rank of the symmetric matrix $Q = (q_{i,j})$.

1.2.1 Sylvester method

In [Syl51], J.J. Sylvester proposed a method to decompose a binary form, that is, a homogeneous polynomial in two variables as a sum of powers of linear forms. This method is based on the following theorem.

Theorem 1.2.3 *The binary form $\psi(x_0, x_1) = \sum_{i=0}^{d} \sigma_i \binom{d}{i} x_0^{d-i} x_1^i$ can be decomposed as a sum of r distinct powers of linear forms*

$$\psi = \sum_{k=1}^{r} \omega_k (\alpha_k x_0 + \beta_k x_1)^d$$

iff there exists a polynomial $p(x_0, x_1) := p_0 x_0^r + p_1 x_0^{r-1} x_1 + \cdots + p_r x_1^r$ *s.t.*

$$\begin{bmatrix} \sigma_0 & \sigma_1 & \dots & \sigma_r \\ \sigma_1 & & & \sigma_{r+1} \\ \vdots & & & \vdots \\ \sigma_{d-r} & \dots & \sigma_{d-1} & \sigma_d \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_r \end{bmatrix} = 0$$

and of the form $p = c \prod_{k=1}^r (\beta_k x_0 - \alpha_k x_1)$ *with* $(\alpha_k, \beta_k) \in \mathbb{K}^2 \setminus \{0\}$ *pairwise distinct directions.*

Proof. If $\psi = \sum_{k=1}^r \omega_k (\alpha_k x_0 + \beta_k x_1)^d$ then $\sigma_i = \sum_{k=1}^r \omega_k \alpha_k^{d-i} \beta_k^i$ and for $j = 0, \dots, d-r$

$$\sum_{i=0}^r \sigma_{i+j} p_i = \sum_{i=0}^r \sum_{k=1}^r \omega_k p_i \alpha_k^{d-i-j} \beta_k^{i+j} = \sum_{k=1}^r \omega_k \alpha_k^{d-r} \beta_k^j p(\alpha_k, \beta_k) = 0$$

Conversely, assume that $p = p_0 x_0^r + p_1 x_0^{r-1} x_1 + \cdots + p_r x_1^r = \prod_{k=1}^r (\beta_k x_0 - \alpha_k x_1)$ with $\vec{p} = [p_0, \dots, p_r] \in \ker H_\sigma^{d-r,r}$ where

$$H_\sigma^{d-r,r} = \begin{bmatrix} \sigma_0 & \sigma_1 & \dots & \sigma_r \\ \sigma_1 & & & \sigma_{r+1} \\ \vdots & & & \vdots \\ \sigma_{d-r} & \dots & \sigma_{d-1} & \sigma_d \end{bmatrix}.$$

By a generic change of coordinates in (x_0, x_1), we can assume that $p_r \neq 0$.

As the directions (α_k, β_k) are pairwise distinct, there exists $\omega_1, \dots, \omega_r$ such that

$$\begin{bmatrix} \alpha_1^0 \beta_1^d & \cdots & \alpha_r^0 \beta_r^d \\ \vdots & & \vdots \\ \alpha_1^d \beta_1^0 & \cdots & \alpha_r^d \beta_r^0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_r \end{bmatrix} = \begin{bmatrix} \sigma_0 \\ \vdots \\ \sigma_{r-1} \end{bmatrix}$$

As $\vec{p} \in \ker H_\sigma$, we deduce that

$$\begin{aligned} p_r \sigma_r &= -\sum_{i=0}^{r-1} p_i \sigma_i \\ &= -\sum_{i=0}^{r-1} \sum_{k=1}^r p_i \omega_k \alpha_k^{d-i} \beta_k^i = -\sum_{k=1}^r \omega_k \sum_{i=0}^{r-1} p_i \alpha_k^{d-i} \beta_k^i \\ &= -\sum_{k=1}^r \omega_k \left(p(\alpha_k, \beta_k) - p_r \alpha_k^{d-r} \beta_k^r \right) = p_r \sum_{k=1}^r \omega_k \alpha_k^{d-r} \beta_k^r \end{aligned}$$

and $p_r \neq 0$ implies that $\sigma_r = \sum_{k=1}^r \omega_k \alpha_k^{d-r} \beta_k^r$. By induction on j, using the relations $p_r \sigma_{r+j} = -\sum_{i=0}^{r-1} p_i \sigma_{i+j}$, we prove similarly that

$$\sigma_{r+j} = \sum_{k=1}^r \omega_k \alpha_k^{d-r-j} \beta_k^{r+j}$$

for $j = 0, \dots, d-r$. This implies that $\psi = \sum_{k=1}^{r} \omega_k (\alpha_k x_0 + \beta_k x_1)^d$. $\square$

1.2.2 Apolarity

Definition 1.2.4 (Apolar product) *For* $f = \sum_{|\alpha|=d} f_\alpha \binom{d}{\alpha} \overline{\boldsymbol{x}}^\alpha$, $g = \sum_{|\alpha|=d} g_\alpha \binom{d}{\alpha} \overline{\boldsymbol{x}}^\alpha \in \mathbb{K}[\overline{\boldsymbol{x}}]_d$,

$$\langle f, g \rangle_d = \sum_{|\alpha|=d} f_\alpha\, g_\alpha \binom{d}{\alpha}.$$

Proposition 1.2.5 $\langle f, (\xi_0 x_0 + \cdots + \xi_n x_n)^d \rangle_d = f(\xi_0, \dots, \xi_n)$.

Given an homogeneous polynomial $\psi \in S_d$, we can define an element $\psi^\star$ of the dual space $S_d{}^\star = \mathrm{Hom}_{\mathbb{K}}(S_d, \mathbb{K})$ as follows

$$\begin{aligned} \psi^\star : S_d &\to \mathbb{K} \\ p &\mapsto \langle \psi, p \rangle_d \end{aligned}$$

By Proposition 1.2.5, $(\xi_0 x_0 + \cdots + \xi_n x_n)^{d\star}$ is the evaluation

$$\begin{aligned} \mathfrak{e}_\xi : S_d &\to \mathbb{K} \\ p &\mapsto p(\xi_0, \dots, \xi_n) \end{aligned}$$

Since the map $\psi \in S_d \to \psi^\star \in S_d{}^\star$ is linear, the tensor decomposition problem can be reformulated as follows:

Problem 1.2.6 (Dual symmetric tensor decomposition)

Given $\psi^\star \in S_d{}^\star$, find a decomposition of ψ of the form

$$\psi^\star = \sum_{i=1}^{r} \omega_i \mathfrak{e}_{\xi_i}$$

for $\xi_i = (\xi_{i,0}, \xi_{i,1}, \dots, \xi_{i,n})$ distinct in $\mathbb{P}(\overline{\mathbb{K}}^{n+1})$, $\omega_i \in \overline{\mathbb{K}}$.

1.2.3 Secants of Veronese variety

Let us give here a geometric view on this decomposition problem.

The evaluation $\mathfrak{e}_\xi \in S_d{}^\star$ at a point $\xi \in \overline{\mathbb{K}}^{n+1}$ is represented in the dual basis of the monomial basis by the vector $(\xi^\alpha)_{|\alpha|=d}$ obtained by evaluation at ξ of the monomials $\overline{\boldsymbol{x}}^\alpha$ which form a basis of S_d. The set of these vectors for non-zero vectors $\xi \in \overline{\mathbb{K}}^{n+1}$ form a projective variety called the *Veronese* variety and denoted hereafter $\mathcal{V}_d^n$. This projective variety is defined by the equations $x_\alpha x_\beta - x_{\alpha'} x_{\beta'} = 0$ for $\alpha, \alpha', \beta, \beta' \in \mathbb{N}^{n+1}$,

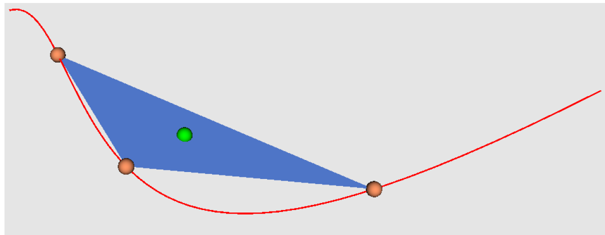


Figure 1.1: A (green) point on the 3^{th} secant of $\mathcal{V}_3^1 \subset \mathbb{P}^3$, which is on the (blue) plane spanned by 3 (orange) points of the (red) curve $\mathcal{V}_3^1$.

$|\alpha| = |\beta| = |\alpha'| = |\beta| = d$ and $\alpha+\beta = \alpha'+\beta'$, where $(x_\alpha)_{|\alpha|=d}$ are the variables associated to the dual basis of the monomial basis. These equations are 2×2 minors of Hankel matrices (see Section 4.1), defining Hankel operators of rank 1.

The dual ψ^* of a tensor, which decomposes as $\psi^\star = \sum_{i=1}^r \omega_i \mathfrak{e}_{\xi_i}$ corresponds to a point in $\mathbb{P}(S_d{}^\star)$, which is in the linear span of the evaluations $[\mathfrak{e}_{\xi_i}] \in \mathcal{V}_d^n$, $i = 1, \ldots, r$. Let

$$S_r^o(\mathcal{V}_d^n) = \{[\psi^\star] \in \mathbb{P}(S_d{}^\star) \mid \psi^\star = \sum_{i=1}^r \omega_i e_i \text{ with } [e_i] \in \mathcal{V}_d^n, \omega_i \in \mathbb{K}\}$$

be the set of points in the linear span of r distinct points of $\mathcal{V}_d^n$. The closure $S_r(\mathcal{V}_d^n) = \overline{S_r^o(\mathcal{V}_d^n)}$ is called the r^{th}-*secant* of $\mathcal{V}_d^n$.

1.3 Multilinear tensor decomposition

A multilinear tensor τ is an element of a space $E_1 \otimes \cdots \otimes E_m$ where E_i are $\mathbb{K}$ vector spaces of dimension $n_i + 1$. Fixing bases of E_i, τ is represented by a multi-index array $(\tau_{i_1,\ldots,i_m})_{0 \leqslant i_l \leqslant n_l} \in \mathbb{K}^{(n_1+1)\times\cdots\times(n_m+1)}$. Equivalently, τ can be represented by a multilinear polynomial

$$\tau(\overline{\boldsymbol{x}}_1, \ldots, \overline{\boldsymbol{x}}_m) = \sum_{0 \leqslant i_l \leqslant n_l} t_{i_1,\ldots,t_m} x_{1,i_1} \cdots x_{m,i_m}$$

in the variables $\overline{\boldsymbol{x}}_j = (x_{j,0}, \ldots, x_{j,n_j})$, $j = 1, \ldots, m$. Let $S_{1,\ldots,1}^{n_1,\ldots,n_m}$ be the vector space of multilinear polynomials in the variables $\overline{\boldsymbol{x}}_1, \ldots, \overline{\boldsymbol{x}}_m$ with coefficients in $\mathbb{K}$.

Problem 1.3.1 (Multilinear tensor decomposition)

Given a multilinear tensor

$$\tau(\overline{\boldsymbol{x}}_1, \ldots, \overline{\boldsymbol{x}}_m) = \sum_{0 \leqslant i_l \leqslant n_l} t_{i_1,\ldots,t_m} x_{1,i_1} \cdots x_{m,i_m}$$

find vectors $\xi_1^i \in \overline{\mathbb{K}}^{n_1+1}, \dots, \xi_m^i \in \overline{\mathbb{K}}^{n_m+1}$ such that

$$\tau(\overline{\boldsymbol{x}}_1, \dots, \overline{\boldsymbol{x}}_m) = \sum_{i=1}^{r} \prod_{j=1}^{m} (\xi_{j,0}^i x_{j,0} + \cdots + \xi_{j,n_j}^i x_{j,n_j})$$

This decomposition means that τ is the sum of the tensor products of the vectors ξ_j^i in $E_1 \otimes \cdots \otimes E_m$:

$$\tau = \sum_{i=1}^{r} \xi_1^i \otimes \cdots \otimes \xi_m^i.$$

The minimal number of terms in such a decomposition is called the *rank* of τ.

Notice that we don't put a weight ω_i in front of each term $\xi_1^i \otimes \cdots \otimes \xi_m^i$ since it can be integrated in the term by scaling one of the vectors ξ_l^i by ω_i. However, in some cases, for instance when the vectors ξ_l^i are normalized, we will introduce these weights in the decomposition: $\tau = \sum_{i=1}^{r} \omega_i \, \xi_1^i \otimes \cdots \otimes \xi_m^i$.

1.3.1 Apolarity

Similarly to symmetric tensors, an apolar product can be defined on multilinear tensors as follows.

Definition 1.3.2 *For all* $\tau = \sum_{0 \leqslant i_l \leqslant n_l} \tau_{i_1,\dots,i_m} x_{1,i_1} \cdots x_{m,i_m}$, $\tau' = \sum_{0 \leqslant i_l \leqslant n_l} \tau'_{i_1,\dots,i_m} x_{1,i_1} \cdots x_{m,i_m} \in S_{1,\dots,1}^{n_1,\dots,n_m}$,

$$\langle \tau, \tau' \rangle = \sum_{0 \leqslant i_l \leqslant n_l} \tau_{i_1,\dots,i_m} \tau'_{i_1,\dots,i_m}.$$

Proposition 1.3.3 *For* $\tau \in S_{1,\dots,1}^{n_1,\dots,n_l}$, $\xi_j = (\xi_{j,0}, \dots, \xi_{j,n_j}) \in \overline{\mathbb{K}}^{n_j+1}$ *for* $j = 1, \dots, m$,

$$\langle \tau, \prod_{j=1}^{m} (\xi_{j,0} x_{j,0} + \cdots + \xi_{j,m} x_{j,m}) \rangle = \tau(\xi_1, \dots, \xi_m).$$

1.3.2 Dual tensor decomposition

For a tensor $\psi = \xi_1 \otimes \cdots \otimes \xi_m$ of rank 1 or equivalently a multilinear polynomial $\psi = \prod_{j=1}^{m} (\xi_{j,0} x_{j,0} + \cdots + \xi_{j,m} x_{j,m})$, and any $f \in S_{1,\dots,1}^{n_1,\dots,n_m}$ we have

$$\langle \psi, f \rangle = f(\xi_1, \dots, \xi_m)$$

Therefore, $\psi^\star : f \in S_{1,\dots,1}^{n_1,\dots,n_m} \to \langle \psi, f \rangle \in \mathbb{K}$ is such that

$$\psi^\star = \mathfrak{e}_\xi \text{ on } S_{1,\dots,1}^{n_1,\dots,n_m}$$

where $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{K}^{n_1+\cdots+n_m}$.

Using this duality, we can reformulation the decomposition problem as follows:

Given $\psi^\star \in (S_{1,\dots,1}^{n_1,\dots,n_m})^\star$, find a decomposition of $\psi^\star$ of the form

$$\psi^\star = \sum_{i=1}^{r} \omega_i \mathfrak{e}_{\xi_i}$$

for distinct directions $\xi_i \in \overline{\mathbb{K}}^{n_1+\dots+n_m}$ and $\omega_i \in \overline{\mathbb{K}} \setminus \{0\}$.

1.3.3 Secant of Segre variety

The set of tensors $\psi = \xi_1 \otimes \cdots \otimes \xi_m \neq 0$ of rank 1 is an algebraic variety of $\mathbb{P}^{n_1+\dots+n_m-1}$, called the *Segre variety*. We denote it by $\mathscr{S}^{n_1,\dots,n_m}$.

Decomposing a multilinear tensor τ as

$$\tau = \sum_{i=1}^{r} \omega_i \, \xi_1^i \otimes \cdots \otimes \xi_m^i$$

means we write τ as an point of the linear span of r points $\xi^i = \xi_1^i \otimes \cdots \otimes \xi_m^i \in \mathscr{S}^{n_1,\dots,n_m}$.

The closure of the points in the linear span of r linearly indenpendent points $\xi_i \in \mathscr{S}^{n_1,\dots,n_m}$ is called the r-th *secant variety* of $\mathscr{S}^{n_1,\dots,n_m}$.

1.4 Simultaneous decomposition

The problem of simultaneous decomposition of a set of tensors consists in finding common points, such that all the tensors can be decomposed in terms of these points. We illustrate it here for symmetric tensors.

Problem 1.4.1 (Simultaneous symmetric tensor decomposition)

Given symmetric tensors $\psi_1, \dots, \psi_m$ of order $d_1, \dots, d_m$, find a simultaneous decomposition of the form

$$\psi_l = \sum_{i=1}^{r} \omega_{i,l} (\xi_{i,0} x_0 + \xi_{i,1} x_1 + \cdots + \xi_{i,n} x_n)^{d_l}$$

where $\xi_l = (\xi_{l,0}, \dots, \xi_{l,n})$ span distinct lines in $\overline{\mathbb{K}}^{n+1}$ and $\omega_{i,l} \in \overline{\mathbb{K}}$ for $l = 1, \dots, m$.

1.4.1 Binary forms

A method similar to Sylvester method can be applied for the simultaneous decomposition of binary forms, based on the following proposition (similar to Theorem 1.2.3).

Proposition 1.4.2 *Let* $\psi_l = \sum_{i=0}^{d_l} \sigma_{1,i}\binom{d_l}{i} x_0^{d_l-i} x_1^i \in \mathbb{K}[x_0, x_1]_{d_l}$ *for* $l = 1, \ldots, m$.

If there exists a polynomial $p(x_0, x_1) := p_0 x_0^r + p_1 x_0^{r-1} x_1 + \cdots + p_r x_1^r$ *such that*

$$\left[\begin{array}{cccc}
\sigma_{1,0} & \sigma_{1,1} & \cdots & \sigma_{1,r} \\
\sigma_{1,1} & & & \sigma_{1,r+1} \\
\vdots & & & \vdots \\
\sigma_{1,d_1-r} & \cdots & \sigma_{1,d_1-1} & \sigma_{1,d_1} \\
\hline
\vdots & & & \vdots \\
\hline
\sigma_{m,0} & \sigma_{m,1} & \cdots & \sigma_{m,r} \\
\sigma_{m,1} & & & \sigma_{r+1} \\
\vdots & & & \vdots \\
\sigma_{m,d_m-r} & \cdots & \sigma_{m,d_m-1} & \sigma_{m,d_m}
\end{array}\right]
\begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_r \end{bmatrix} = 0$$

and of the form $p = c\prod_{k=1}^{r}(\beta_k x_0 - \alpha_k x_1)$ *with* $(\alpha_k, \beta_k) \in \mathbb{K}^2$ *pairewise distinct directions, then*

$$\psi_l = \sum_{i=1}^{r} \omega_{i,l}(\alpha_l x_0 + \beta_l x_1)^{d_l}$$

for $\omega_{i,l} \in \overline{\mathbb{K}}$ *and* $l = 1, \ldots, m$.

Proof. We use the same proof as for Theorem 1.2.3 applied to the Hankel block associated to σ_i (for $i = 1, \ldots, m$). $\square$

1.5 Sparse interpolation

Given a black-box polynomial function $f(\boldsymbol{x})$

find $r \in \mathbb{N}, \omega_i \in \mathbb{C}, \alpha_i \in \mathbb{N}^n$ such that $f(\boldsymbol{x}) = \sum_{i=1}^{r} \omega_i \boldsymbol{x}^{\alpha_i}$.

Example 1.5.1 *Consider the polynomial* $f(x_1, x_2) = x_1^{33} x_2^{12} - 5\, x_1 x_2^{25} + 101$.

Let us choose $\varphi_1 = e^{\frac{6i\pi}{100}}, \varphi_2 = e^{\frac{14i\pi}{100}}$. *We evaluate* f *at the points* $(\varphi_1^k, \varphi_2^k)$ *for* $k = 0, \ldots, d$ *and get the sequence*

$$\sigma_k = f(\varphi_1^k, \varphi_2^k) = (\varphi_1^{33}\varphi_2^{12})^k - 5(\varphi_1 \varphi_2^{25}) + 101 = \xi_1^k - 5\xi_2^k + 101\xi_3^k,$$

where $\xi_1 = \varphi_1^{33}\varphi_2^{12} = e^{47\frac{2\pi}{100}}$, $\xi_2 = \varphi_1\varphi_2^{25} = e^{78\frac{2\pi}{100}}$, $\xi_3 = 1$. *We compute a non-zero element* $[p_0, p_1, \ldots, p_r]$ *in the kernel of the matrix*

$$H_\sigma = \begin{bmatrix} \sigma_0 & \sigma_1 & \ldots & \sigma_r \\ \sigma_1 & & & \sigma_{r+1} \\ \vdots & & & \vdots \\ \sigma_{d-r} & \ldots & \sigma_{d-1} & \sigma_d \end{bmatrix}$$

for $r = 3$ *and* $d \geq 5$. *The roots of the polynomial* $p(x) := p_0 + p_1 x + \cdots + p_r x^r$ *are* $\xi_1, \xi_2, \xi_3 \in \mathbb{C}$. *Computing* $m_i = -\mathbf{i}\frac{100}{2\pi}\log(\xi_i)$, *we get*

$$\begin{aligned} m_1 &= 47 = 3 \times 33 + 7 \times 12 - 100 \times 7 \\ m_2 &= 78 = 3 \times 1 + 7 \times 25 - 100 \times 1 \\ m_3 &= 0 \end{aligned}$$

Decomposing $m_i = 3\,a_i + 5\,b_i + 100\,c_i$ *modulo* $3 \times 7 \times 100 = 2100$, *we recover the exponents of the terms of* f: $(33, 12)$, $(1, 25)$, $(0, 0)$.

The coefficients ω_i *are recovered by solving the system*

$$\begin{bmatrix} 1 & 1 & 1 \\ \xi_1 & \xi_2 & \xi_3 \\ \xi_1^2 & \xi_2^2 & \xi_3^2 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \sigma_2 \end{bmatrix}$$

The solution of this system yields $(\omega_1, \omega_2, \omega_3) = (1, -5, 101)$.

Chapter 2

Duality

In this chapter, we consider polynomials and series with coefficients in a field $\mathbb{K}$ of characteristic 0. In the applications, we are going to take $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$.

We are going to use the following notation: $\mathbb{K}[x_1, \ldots, x_n] = \mathbb{K}[\boldsymbol{x}] = R$ is the ring of polynomials in the variable $x_1, \ldots, x_n$ with coefficients in the field $\mathbb{K}$, $\mathbb{K}[[y_1, \ldots, y_n]] = \mathbb{K}[[\boldsymbol{y}]]$ is the ring of formal power series in the variables $y_1, \ldots, y_n$ with coefficients in $\mathbb{K}$.

The dual of the ring of polynomials is

$$\mathbb{K}[\boldsymbol{x}]^\star = \{\sigma : \mathbb{K}[\boldsymbol{x}] \to \mathbb{K} \text{ linear}\} = \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}[\boldsymbol{x}], \mathbb{K}).$$

Given $\sigma \in \mathbb{K}[\boldsymbol{x}]^\star$, $p \in \mathbb{K}[\boldsymbol{x}]$, we denote by $\langle \sigma \mid p \rangle$ the value of σ applied to p. The elements in $\mathbb{K}[\boldsymbol{x}]^\star$ will also be called *linear functionals* on $\mathbb{K}[\boldsymbol{x}]$.

For any $\sigma \in \mathbb{K}[\boldsymbol{x}]^\star$, the inner product associated to σ on $\mathbb{K}[\boldsymbol{x}]$ is defined as follows:

$$\begin{array}{rcl} \mathbb{K}[\boldsymbol{x}] \times \mathbb{K}[\boldsymbol{x}] & \to & \mathbb{K} \\ (p,q) & \mapsto & \langle p,q \rangle_\sigma := \langle \sigma \mid p\, q \rangle. \end{array}$$

The dual space $\mathbb{K}[\boldsymbol{x}]^\star$ has a natural structure of $\mathbb{K}[\boldsymbol{x}]$-module, defined as follows: $\forall \sigma \in \mathbb{K}[\boldsymbol{x}]^\star, \forall p, q \in \mathbb{K}[\boldsymbol{x}]$,

$$\langle p \star \sigma \mid q \rangle \quad = \quad \langle \sigma \mid p\, q \rangle.$$

The operator $\sigma \in \mathbb{K}[\boldsymbol{x}]^\star \mapsto p \star \sigma \in \mathbb{K}[\boldsymbol{x}]^\star$ is, by definition, the transpose or adjoint of the multiplication by p: $q \in \mathbb{K}[\boldsymbol{x}] \mapsto p\, q \in \mathbb{K}[\boldsymbol{x}]$.

We check that $\forall \sigma \in \mathbb{K}[[\boldsymbol{y}]], \forall p, q \in \mathbb{K}[\boldsymbol{x}]$, $(pq) \star \sigma = p \star (q \star \sigma)$. See e.g. [Ems78], [EM07a] for more details.

2.1 Sequences

To describe a linear functional $\sigma \in \mathbb{K}[\boldsymbol{x}]^*$, it is enough to know it on a basis of $\mathbb{K}[\boldsymbol{x}]$. A natural basis is the monomial basis $(\boldsymbol{x}^\alpha)_{\alpha \in \mathbb{N}^n}$. The linear functional σ is uniquely defined by the sequence

$$(\langle \sigma \mid \boldsymbol{x}^\alpha \rangle)_{\alpha \in \mathbb{N}^n}.$$

The values $\sigma_\alpha := \langle \sigma \mid \boldsymbol{x}^\alpha \rangle$ for $\alpha \in \mathbb{N}^n$ are called the *moments* of σ.

Given a polynomial $p = \sum_{\alpha \in A} p_\alpha \boldsymbol{x}^\alpha \in \mathbb{K}[\boldsymbol{x}]$ with $A \subset \mathbb{N}^n$ finite, the value of σ applied to p is by linearity

$$\langle \sigma \mid p \rangle = \sum_{\alpha \in A} p_\alpha \sigma_\alpha.$$

This allows us to identify $\mathbb{K}[\boldsymbol{x}]^*$ with the vector space of multi-index sequences $\mathbb{K}^{\mathbb{N}^n}$ via the isomorphism:

$$\begin{array}{rcl} \mathfrak{i}_0 : \mathbb{K}[\boldsymbol{x}]^* & \to & \mathbb{K}^{\mathbb{N}^n} \\ \sigma & \mapsto & (\langle \sigma \mid \boldsymbol{x}^\alpha \rangle)_{\alpha \in \mathbb{N}^n} \end{array} \tag{2.1}$$

More generally, choosing a point $\zeta \in \mathbb{K}^n$ and monomials $((\boldsymbol{x} - \zeta)^\alpha)_{\alpha \in \mathbb{N}^n}$ as a basis of $\mathbb{K}[\boldsymbol{x}]$, we define the isomorphism $\mathfrak{i}_\zeta : \sigma \in \mathbb{K}[\boldsymbol{x}]^* \mapsto (\langle \sigma \mid (\boldsymbol{x} - \zeta)^\alpha \rangle)_{\alpha \in \mathbb{N}^n}$.

The structure of $\mathbb{K}[\boldsymbol{x}]$-module of $\mathbb{K}[\boldsymbol{x}]^*$ in this representation is given by shift operators. Let $\mathscr{S}_i : (\sigma_\alpha)_{\alpha \in \mathbb{N}^n} \in \mathbb{K}^{\mathbb{N}^n} \mapsto (\sigma_{e_i + \alpha})_{\alpha \in \mathbb{N}^n} \in \mathbb{K}^{\mathbb{N}^n}$ be the *shift operator* by e_i, where $(e_i)_{i=1,\dots,n}$ is the canonical basis of $\mathbb{N}^n$. Then, we have

$$x_i \star \sigma = (\langle \sigma \mid x_i \boldsymbol{x}^\alpha \rangle)_{\alpha \in \mathbb{N}^n} = (\langle \sigma \mid \boldsymbol{x}^{e_i + \alpha} \rangle)_{\alpha \in \mathbb{N}^n} = \mathscr{S}_i(\sigma).$$

More generally, for $p \in \mathbb{K}[\boldsymbol{x}]$, $p \star \sigma = p(\mathscr{S}_1, \dots, \mathscr{S}_n)(\sigma)$.

For $p = \sum_\beta p_\beta \boldsymbol{x}^\beta \in \mathbb{K}[\boldsymbol{x}]$ and $\sigma = \sum_{\alpha \in \mathbb{N}^n} \sigma_\alpha \frac{\boldsymbol{y}^\alpha}{\alpha!} \in \mathbb{K}[[\boldsymbol{y}]]$, the series expansion of $p \star \sigma$ is $p \star \sigma = (\rho_\alpha)_{\alpha \in \mathbb{N}^n}$ with $\forall \alpha \in \mathbb{N}^n$,

$$\rho_\alpha = \sum_\beta p_\beta \sigma_{\alpha + \beta}.$$

Identifying $\mathbb{K}[\boldsymbol{x}]$ with the set $\ell_0(\mathbb{K}^{\mathbb{N}^n})$ of sequences $p = (p_\alpha)_{\alpha \in \mathbb{N}^n}$ of finite support (i.e. a finite number of non-zero terms), we see that $p \star \sigma$ is the *cross-correlation* sequence of p and σ.

2.2 Taylor series

Assume here that char $\mathbb{K} = 0$. Elements in $\mathbb{K}[\boldsymbol{x}]^\star$ can also be represented by formal power series, using following the natural isomorphism between the ring of formal power series and the dual of $\mathbb{K}[\boldsymbol{x}]$:

$$\begin{array}{rcl} \mathbb{K}[[y_1,\ldots,y_n]] \times \mathbb{K}[x_1,\ldots,x_n] & \to & \mathbb{K} \\ (\boldsymbol{y}^\alpha, \boldsymbol{x}^\beta) & \mapsto & \langle \boldsymbol{y}^\alpha \mid \boldsymbol{x}^\beta \rangle = \left\{ \begin{array}{ll} \alpha! & \text{if } \alpha = \beta \\ 0 & \text{otherwise}. \end{array} \right. \end{array}$$

Namely, if $\sigma \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}[\boldsymbol{x}], \mathbb{K}) = R^*$ is an element of the dual of $\mathbb{K}[\boldsymbol{x}]$, it can be represented by the series:

$$\begin{array}{rcl} \iota_0 : \mathbb{K}[\boldsymbol{x}]^\star & \to & \mathbb{K}[[\boldsymbol{y}]] \\ \sigma & \mapsto & \sigma(\boldsymbol{y}) = \displaystyle\sum_{\alpha \in \mathbb{N}^n} \sigma(\boldsymbol{x}^\alpha) \frac{\boldsymbol{y}^\alpha}{\alpha!}, \end{array} \tag{2.2}$$

so that we have $\langle \sigma(\boldsymbol{y}) | \boldsymbol{x}^\alpha \rangle = \sigma(\boldsymbol{x}^\alpha)$. The map $\sigma \in R^* \mapsto \sum_{\alpha \in \mathbb{N}^n} \sigma(\boldsymbol{x}^\alpha) \frac{\boldsymbol{y}^\alpha}{\alpha!} \in \mathbb{K}[[\boldsymbol{y}]]$ is an isomorphism and any series $\sigma(\boldsymbol{y}) = \sum_{\alpha \in \mathbb{N}^n} \sigma_\alpha \frac{\boldsymbol{y}^\alpha}{\alpha!} \in \mathbb{K}[[\boldsymbol{y}]]$ can be interpreted as a linear form

$$p = \sum_{\alpha \in A \subset \mathbb{N}^n} p_\alpha \boldsymbol{x}^\alpha \in \mathbb{K}[\boldsymbol{x}] \mapsto \langle \sigma \mid p \rangle = \sum_{\alpha \in A \subset \mathbb{N}^n} p_\alpha \sigma_\alpha.$$

Any linear form $\sigma \in R^\star$ is uniquely defined by its coefficients $\sigma_\alpha = \langle \sigma \mid \boldsymbol{x}^\alpha \rangle$ for $\alpha \in \mathbb{N}^n$, which are called the *moments* of σ.

From now on, we identify the dual $R^\star = \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}[\boldsymbol{x}], \mathbb{K})$ with $\mathbb{K}[[\boldsymbol{y}]]$. Using this identification, the dual basis of the monomial basis $(\boldsymbol{x}^\alpha)_{\alpha \in \mathbb{N}^n}$ is $\left(\frac{\boldsymbol{y}^\alpha}{\alpha!}\right)_{\alpha \in \mathbb{N}^n}$.

If $\mathbb{K}$ is a subfield of a field $\mathbb{L}$, we have the embedding $\mathbb{K}[[\boldsymbol{y}]] \hookrightarrow \mathbb{L}[[\boldsymbol{y}]]$, which allows to identify an element of $\mathbb{K}[\boldsymbol{x}]^*$ with an element of $\mathbb{L}[\boldsymbol{x}]^*$.

The truncation of an element $\sigma(\boldsymbol{y}) = \sum_{\alpha \in \mathbb{N}^n} \sigma_\alpha \frac{\boldsymbol{y}^\alpha}{\alpha!} \in \mathbb{K}[[\boldsymbol{y}]]$ in degree d is $\sum_{|\alpha| \leqslant d} \sigma_\alpha \frac{\boldsymbol{y}^\alpha}{\alpha!}$. It is denoted $\sigma(\boldsymbol{y}) + \mathcal{O}(\boldsymbol{y})^{d+1}$, that is, the class of σ modulo the ideal $(y_1, \ldots, y_n)^{d+1} \subset \mathbb{K}[[\boldsymbol{y}]]$.

The structure $\mathbb{K}[\boldsymbol{x}]$-module of $\mathbb{K}[\boldsymbol{x}]^\star$ is given as follows.

Lemma 2.2.1 $\forall p \in \mathbb{K}[\boldsymbol{x}], \forall \sigma \in \mathbb{K}[[\boldsymbol{y}]]$, $p(\boldsymbol{x}) \star \sigma(\boldsymbol{y}) = p(\partial_{y_1}, \ldots, \partial_{y_n})(\sigma)$.

Proof. We first prove the relation for $p = x_i$ ($i \in [1, n]$) and $\sigma = \boldsymbol{y}^\alpha$ ($\alpha \in \mathbb{N}^n$). Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the exponent vector of x_i. $\forall \beta \in \mathbb{N}^n$ and $\forall i \in [1, n]$, we have

$$\begin{array}{rcl} \langle x_i \star \boldsymbol{y}^\alpha | \boldsymbol{x}^\beta \rangle & = & \langle \boldsymbol{y}^\alpha | x_i \boldsymbol{x}^\beta \rangle = \quad \alpha! \quad \text{if} \quad \alpha = \beta + e_i \quad \text{and} \quad 0 \quad \text{otherwise} \\ & = & \alpha_i \langle \boldsymbol{y}^{\alpha - e_i} | \boldsymbol{x}^\beta \rangle. \end{array}$$

with the convention that $\boldsymbol{y}^{\alpha - e_i} = 0$ if $\alpha_i = 0$. This shows that $x_i \star \boldsymbol{y}^\alpha = \alpha_i \boldsymbol{y}^{\alpha - e_i} = \partial_{y_i}(\boldsymbol{y}^\alpha)$. By transitivity and bilinearity of the product $\star$, we deduce that $\forall p \in \mathbb{K}[\boldsymbol{x}], \forall \sigma \in \mathbb{K}[[\boldsymbol{y}]]$,

$p(\boldsymbol{x}) \star \sigma(\boldsymbol{y}) = p(\partial_{y_1}, \ldots, \partial_{y_n})(\sigma).$ □

This property can be useful to analyze the solution of partial differential equations. Let $p_1(\partial_{y_1}, \ldots, \partial_{y_n}), \ldots, p_s(\partial_{y_1}, \ldots, \partial_{y_n}) \in \mathbb{K}[\partial_1, \ldots, \partial_n]$ be a set of partial differential polynomials with constant coefficients $\in \mathbb{K}$. The set of solutions $\sigma \in \mathbb{K}[[\boldsymbol{y}]]$ of the system

$$p_1(\partial_{y_1}, \ldots, \partial_{y_n})(\sigma) = 0, \ldots, p_s(\partial_{y_1}, \ldots, \partial_{y_n})(\sigma) = 0$$

is in correspondence with the elements $\sigma \in (p_1, \ldots, p_s)^{\perp}$, which satisfy $p_i \star \sigma = 0$ for $i = 1, \ldots, s$ (see Theorem 3.4.4). The variety $\mathcal{V}(p_1, \ldots, p_n) \subset \overline{\mathbb{K}}^n$ is called the *characteristic variety* and $I = (p_1, \ldots, p_n)$ the *characteristic ideal* of the system of partial differential equations.

2.2.1 Polynomial-Exponential series

Among the elements of $\mathbb{K}[\boldsymbol{x}]^{\star} \equiv \mathbb{K}[[\boldsymbol{y}]]$, we have the evaluations at points of $\mathbb{K}^n$:

Definition 2.2.2 *The evaluation at a point $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{K}^n$ is:*

$$\begin{array}{rcl} \mathfrak{e}_\xi : \mathbb{K}[x_1, \ldots x_n] & \to & \mathbb{K} \\ p(\boldsymbol{x}) & \mapsto & p(\xi) \end{array}$$

It corresponds to the series:

$$\mathfrak{e}_\xi(\boldsymbol{y}) = \sum_{\alpha \in \mathbb{N}^n} \xi^\alpha \frac{\boldsymbol{y}^\alpha}{\alpha!} = e^{\xi_1 y_1 + \cdots + \xi_n y_n} = e^{\langle \xi, \mathbf{y} \rangle}.$$

Using this formalism, the series $\sigma(\boldsymbol{y}) = \sum_{i=1}^{r} \omega_i \mathfrak{e}_{\xi_i}(\boldsymbol{y})$ with $\omega_i \in \mathbb{K}$ can be interpreted as a linear combination of evaluations at the points ξ_i with coefficients $\omega_i \in \mathbb{K}$, for $i = 1, \ldots, r$. These series belong to the more general family of polynomial-exponential series, that we define now.

Definition 2.2.3 *Let*

$$\mathcal{P}ol\mathcal{E}xp(y_1, \ldots, y_n) = \left\{ \sigma = \sum_{i=1}^{r} \omega_i(\boldsymbol{y}) \mathfrak{e}_{\xi_i}(\boldsymbol{y}) \in \mathbb{K}[[\boldsymbol{y}]] \mid \xi_i \in \mathbb{K}^n, \omega_i(\boldsymbol{y}) \in \mathbb{K}[\boldsymbol{y}] \right\}$$

be the set of polynomial-exponential series. The polynomials $\omega_i(\boldsymbol{y})$ are called the weights of σ and ξ_i the frequencies.

Notice that the product of $\boldsymbol{y}^\alpha \mathfrak{e}_\xi(\boldsymbol{y})$ with a monomial $\boldsymbol{x}^\beta \in \mathbb{C}[x_1, \ldots x_n]$ is given by

$$\begin{aligned} \langle \boldsymbol{y}^\alpha \mathfrak{e}_\xi(\boldsymbol{y}) \mid \boldsymbol{x}^\beta \rangle &= \frac{\beta!}{(\beta - \alpha)!} \xi^{\beta - \alpha} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} (\boldsymbol{x}^\beta)(\xi) \text{ if } \alpha_i \leqslant \beta_i \text{ for } i = 1, \ldots, n \qquad (2.3) \\ &= 0 \text{ otherwise}. \end{aligned}$$

Therefore an element $\sigma = \sum_{i=1}^{r} \omega_i(\mathbf{y})\mathfrak{e}_{\xi_i}(\mathbf{y})$ of $\mathscr{P}ol\mathscr{E}xp(\mathbf{y})$ can also be seen as a sum of *polynomial differential operators* $\omega_i(\partial)$ "at" the points ξ_i, that we call *infinitesimal operators*: $\forall p \in \mathbb{K}[\mathbf{x}], \langle\sigma|p\rangle = \sum_{i=1}^{r} \omega_i(\partial)(p)(\xi)$.

Lemma 2.2.4 *The series $\mathbf{y}^{\alpha_{i,j}}\mathfrak{e}_{\xi_i}(\mathbf{y})$ for $i = 1,\ldots,r$ and $j = 1,\ldots\mu_i$ with $\alpha_{i,1},\ldots,\alpha_{i,\mu_i} \in \mathbb{N}^n$ and $\xi_i \in \mathbb{K}^n$ pairwise distinct are linearly independent.*

Proof. Suppose that there exist $w_{i,j} \in \mathbb{K}$ such that $\sigma(\mathbf{y}) = \sum_{i=1}^{r}\sum_{j=1}^{\mu_i} \omega_{i,j}\mathbf{y}^{\alpha_{i,j}}\mathfrak{e}_{\xi_i}(\mathbf{y}) = 0$ and let $\omega_i(\mathbf{y}) = \sum_{j=1}^{\mu_i} \omega_{i,j}\mathbf{y}^{\alpha_{i,j}}$. Then $\forall p \in \mathbb{K}[\mathbf{x}], p \star \sigma = 0 = \sum_{i=1}^{r} p(\xi_i + \partial_y)(\omega_i)\mathfrak{e}_{\xi_i}(\mathbf{y})$. If the weights $\omega_i(\mathbf{y}) \in \mathbb{K}$ are of degree 0, by choosing for p an interpolation polynomial at one of the distinct points ξ_i, we deduce that $\omega_i = 0$ for $i = 1,\ldots,r$. If the weights $\omega_i(\mathbf{y}) \in \mathbb{K}$ are degree $\geqslant 1$, by choosing $p = l(\mathbf{x}) - l(\xi_i) \in \mathbb{K}[\mathbf{x}]$ for a separating polynomial l of degree 1 ($l(\xi_i) \neq l(\xi_j)$ if $i \neq j$), we can reduce to a case where at least one of the non-zero weights has one degree less. By induction on the degree, we deduce that $\omega_i(\mathbf{y}) = 0$ for $i = 1,\ldots,r$. This proves the linear independency of the series $\mathbf{y}^{\alpha_{i,j}}\mathfrak{e}_{\xi_i}(\mathbf{y})$ for any $\alpha_{i,1},\ldots,\alpha_{i,\mu_i} \in \mathbb{N}^n$ and $\xi_i \in \mathbb{K}^n$ pairwise distinct. $\square$

Lemma 2.2.5 $\forall p \in \mathbb{K}[\mathbf{x}], \forall \omega \in \mathbb{K}[[\mathbf{y}]], \xi \in \mathbb{K}^n, p(\mathbf{x}) \star (\omega(\mathbf{y})\mathfrak{e}_\xi(\mathbf{y})) = p(\xi_1 + \partial_{y_1},\ldots,\xi_n + \partial_{y_n})(\omega(\mathbf{y}))\mathfrak{e}_\xi(\mathbf{y})$.

Proof. By Lemma 2.2.1, $x_i \star (\omega(\mathbf{y})\mathfrak{e}_\xi(\mathbf{y})) = \partial_{y_i}(\omega)(\mathbf{y})\mathfrak{e}_\xi(\mathbf{y}) + \xi_i\,\omega(\mathbf{y})\mathfrak{e}_\xi(\mathbf{y}) = (\xi_i + \partial_{y_i})(\omega(\mathbf{y}))\mathfrak{e}_\xi(\mathbf{y})$ for $i = 1,\ldots,n$. We deduce that the relation is true for any polynomial $p \in \mathbb{K}[\mathbf{x}]$ by repeated multiplications by the variables and linear combination. $\square$

2.3 Dual series

For $\mathbb{K}$ of any characteristic, another representation of elements of $\mathbb{K}[\mathbf{x}]^*$ as formal power series, is based on the following isomorphism:

$$\begin{array}{rcl} \iota_0 : \mathbb{K}[\mathbf{x}]^* & \to & \mathbb{K}[[\mathbf{z}]] \\ \sigma & \mapsto & \sigma(\mathbf{z}) = \sum_{\alpha\in\mathbb{N}^n} \langle\sigma \mid \mathbf{z}^\alpha\rangle \mathbf{z}^\alpha \end{array} \tag{2.4}$$

where $\mathbf{z} = (z_1,\ldots,z_n)$ is a set of new variables, Using the following pairing:

$$\begin{array}{rcl} \mathbb{K}[[z_1,\ldots,z_n]] \times \mathbb{K}[x_1,\ldots,x_n] & \to & \mathbb{K} \\ (\mathbf{z}^\alpha, \mathbf{x}^\beta) & \mapsto & \langle \mathbf{z}^\alpha \mid \mathbf{x}^\beta\rangle = \left\{ \begin{array}{ll} 1 & \text{if } \alpha = \beta \\ 0 & \text{otherwise}. \end{array} \right. \end{array}$$

we have, for any $p = \sum_{\alpha\in A\subset\mathbb{N}^n} p_\alpha \mathbf{x}^\alpha$ with A finite and any $\sigma \in \mathbb{K}[\mathbf{x}]^*$,

$$\langle\sigma(\mathbf{z}) \mid p\rangle = \sum_{\alpha\in A} \sigma_\alpha p_\alpha.$$

In this representation, $(\boldsymbol{z}^{\alpha})_{\alpha\in\mathbb{N}^n}$ is the basis of $\mathbb{K}[\boldsymbol{x}]^{\star}$ dual to the monomial basis $(\boldsymbol{x}^{\alpha})_{\alpha\in\mathbb{N}^n}$ of $\mathbb{K}[\boldsymbol{x}]$.

The map, which associates to the sequence $(\sigma_{\alpha})_{\alpha\in\mathbb{N}^n}$, the formal power eries $\sigma(\boldsymbol{z}) = \sum_{\alpha\in\mathbb{N}^n}\sigma_{\alpha}\,\boldsymbol{z}^{\alpha}$ is the *Z-transform* of the sequence $(\sigma(\boldsymbol{x}^{\alpha}))_{\alpha\in\mathbb{N}^n}$ [enc16]. It corresponds to the embedding in the ring of *divided powers* $(\boldsymbol{z}^{\alpha} = \frac{y^{\alpha}}{\alpha!})$ [Eis94][Sec. A 2.4], [IK99][Appendix A]. It allows to extend the duality properties to any field $\mathbb{K}$, which is not of characteristic 0.

The transformation of the series $\sum_{\alpha\in\mathbb{N}^n}\sigma_{\alpha}\boldsymbol{z}^{\alpha}\in\mathbb{K}[[\boldsymbol{z}]]$ into the series $\sum_{\alpha\in\mathbb{N}^n}\sigma_{\alpha}\frac{y^{\alpha}}{\alpha!}\in\mathbb{K}[[\boldsymbol{y}]]$ is known as the *Borel transform* [enc16].

In this representation, the structure of $\mathbb{K}[\boldsymbol{x}]$-module of $\mathbb{K}[\boldsymbol{x}]^{\star}$ is described as follow. For $\alpha,\beta\in\mathbb{N}^n$, we have $\boldsymbol{x}^{\alpha}\star\boldsymbol{z}^{\beta} = \pi_{+}(\boldsymbol{z}^{\beta-\alpha})$ where π_{+} is projection on the formal power series with positive exponents, which are spanned by the monomials $\boldsymbol{z}^{\alpha}$ with $\alpha\in\mathbb{N}^n$. For $\alpha\in\mathbb{Z}^n$, $\pi_{+}(\boldsymbol{z}^{\alpha}) = \begin{cases} \boldsymbol{z}^{\alpha} \text{ if } \alpha\in\mathbb{N}^n, \\ 0 \text{ otherwise.}\end{cases}$
More generally, for any $p\in\mathbb{K}[\boldsymbol{x}], \sigma\in\mathbb{K}[\boldsymbol{x}]^{\star}$,

$$p\star\sigma = \pi_{+}(p(z_1^{-1},\dots,z_n^{-1})\sigma(\boldsymbol{z})).$$

In this representation, z_i plays the role of the inverse of x_i. This explains the terminology of inverse system, introduced in [Mac16].

With this formalism, the variables $x_1,\dots,x_n$ act on the series in $\mathbb{K}[[\boldsymbol{z}]]$ as shift operators:

$$x_i\star\left(\sum_{\alpha\in\mathbb{N}^n}\sigma_{\alpha}\boldsymbol{z}^{\alpha}\right) = \sum_{\alpha\in\mathbb{N}^n}\sigma_{\alpha+e_i}\boldsymbol{z}^{\alpha}$$

where $e_1,\dots,e_n$ is the canonical basis of $\mathbb{N}^n$. Therefore, for any $p_1,\dots,p_n\in\mathbb{K}[\boldsymbol{x}]$, the system of equations

$$p_1\star\sigma = 0,\dots,p_n\star\sigma = 0$$

corresponds to a system of *difference equations* on $\sigma\in\mathbb{K}[[\boldsymbol{z}]]$.

2.3.1 Rational series

In this setting, the evaluation $\mathfrak{e}_{\xi}$ at a point $\xi\in\mathbb{K}^n$ is represented in $\mathbb{K}[[\boldsymbol{z}]]$ by the rational fraction $\frac{1}{\prod_{j=1}^{n}(1-\xi_j z_j)}$. The series $\boldsymbol{y}^{\beta}\mathfrak{e}_{\xi}\in\mathbb{K}[[\boldsymbol{y}]]$ corresponds to the series of $\mathbb{K}[[\boldsymbol{z}]]$

$$\sum_{\alpha\in\mathbb{N}^n}\frac{(\alpha+\beta)!}{\alpha!}\xi^{\alpha}\boldsymbol{z}^{\alpha+\beta} = \beta!\boldsymbol{z}^{\beta}\sum_{\alpha\in\mathbb{N}^n}\binom{\alpha+\beta}{\beta}\xi^{\alpha}\boldsymbol{z}^{\alpha} = \frac{\beta!\boldsymbol{z}^{\beta}}{\prod_{j=1}^{n}(1-\xi_j z_j)^{1+\beta_j}}.$$

The reconstruction of truncated series consists then in finding points $\xi_1,\dots,\xi_{r'}\in\mathbb{K}^n$ and finite sets A_i of coefficients $\omega_{i,\alpha}\in\mathbb{K}$ for $i=1,\dots,r'$ and $\alpha\in A_i$ such that

$$\sum_{\alpha\in\mathbb{N}^n}\sigma_{\alpha}\boldsymbol{z}^{\alpha} = \sum_{i=1}^{r'}\sum_{\alpha\in A_i}\frac{\omega_{i,\alpha}\boldsymbol{z}^{\alpha}}{\prod_{i=1}^{n}(1-\xi_{i,j}z_j)^{1+\alpha_j}} = \prod_{i=1}^{n}\bar{z}_j\sum_{i=1}^{r'}\sum_{\alpha\in A_i}\frac{\omega_{i,\alpha}}{\prod_{i=1}^{n}(\bar{z}_j-\xi_{i,j})^{1+\alpha_j}} \tag{2.5}$$

where $\bar{z}_j = z_j^{-1}$.

In the univariate case, this reduces to computing polynomials $\omega(z), \delta(z) = \prod_{i=1}^{r'}(1-\xi_i z)^{\mu_i} \in \mathbb{K}[z]$ with $\deg(\omega) < \deg(\delta) = \sum_i \mu_i = r$ such that

$$\sum_{k\in\mathbb{N}} \sigma_k z^k = \frac{w(z)}{\delta(z)}.$$

The decomposition can thus be computed from the *Padé approximant* of order $(r-1, r)$ of the sequence $(\sigma_k)_{k\in\mathbb{N}}$ (see e.g. [vzGG13][chap. 5]).

Unfortunately, this representation in terms of Padé approximant does not extend so nicely to the multivariate case. The series $\sigma = \sum_{\alpha\in\mathbb{N}^n} \sigma_\alpha \boldsymbol{z}^\alpha$ with a decomposition of the form (2.5) correspond to the series $\sum_{\alpha\in\mathbb{N}^n} \sigma_\alpha \boldsymbol{z}^{-\alpha}$, which is rational of the form $\frac{\boldsymbol{z}^1 p(\boldsymbol{z})}{\prod q_i(z_i)}$ with a splitable denominator where $\deg(q_i) \geqslant 1$ are univariate polynomials (see e.g. [Pow82], [Bar84]). Though Padé approximants could be computed in this case by "separating" the variables (or by relaxing the constraints on the Padé approximants [Cuy99]), the rational fraction $\frac{\boldsymbol{z}^1 p(\boldsymbol{z})}{\prod q_i(z_i)}$ is mixing the coordinates of the points $\xi_1, \ldots, \xi_{r'} \in \mathbb{K}^n$ and the weights $\omega_{i,\alpha}$.

As the duality between multiplication and differential operators is less natural in $\mathbb{K}[[\boldsymbol{z}]]$, we will use hereafter the identification (2.2) of R^* with $\mathbb{K}[[\boldsymbol{y}]]$, when $\mathbb{K}$ is of characteristic 0.

2.4 Inverse systems

For a vector space $V \subset \mathbb{K}[\boldsymbol{x}]$, let $V^\perp = \{\sigma \in \mathbb{K}[\boldsymbol{x}]^\star \mid \langle \sigma \mid v\rangle = 0, \forall v \in V\}$. Similarly, for a vector space $D \subset \mathbb{K}[\boldsymbol{x}]^\star$, $D^\perp = \{p \in \mathbb{K}[\boldsymbol{x}] \mid \langle \delta \mid p\rangle = 0, \forall \delta \in D\}$.

The set of formal power series $\mathbb{K}[[\boldsymbol{y}]]$ is a topological space for the $\boldsymbol{m}$-adic topology where $\boldsymbol{m} = (y_1, \ldots, y_n)$. The dual space $\mathbb{K}[\boldsymbol{x}]^\star$, equipped with topology of simple convergence is also a topological space. For these topologies, the isomorphism (2.2) between $\mathbb{K}[[\boldsymbol{y}]]$ and $\mathbb{K}[\boldsymbol{x}]^\star$ is an isomorphism of topological vector spaces. In particular, $D \subset \mathbb{K}[\boldsymbol{x}]^\star$ is closed iff $D^{\perp\perp} = D$.

Given an ideal $I \subset \mathbb{K}[\boldsymbol{x}]$, stable by multiplication by $x_i \in \mathbb{K}[\boldsymbol{x}]$, $i = 1, \ldots, n$, the orthogonal $I^\perp$ is stable by the transpose multiplication, which is the derivation (see Lemma 2.2.1):

$$\forall \sigma \in I^\perp, i = 1, \ldots, n, x_i \star \sigma = \partial_{y_i} \sigma \in I^\perp.$$

Thus, the map $I \subset \mathbb{K}[\boldsymbol{x}] \mapsto I^\perp \subset \mathbb{K}[[\boldsymbol{y}]]$ defines a correspondence between the ideals I of $\mathbb{K}[\boldsymbol{x}]$ and the vector spaces of $\mathbb{K}[[\boldsymbol{y}]]$ which are closed for the $\boldsymbol{m}$-adic topology and stable by derivation with respect to y_i. See e.g. [Ems78] for more details.

This leads to the following definitions:

Definition 2.4.1 *For a subset $D \subset \mathbb{K}[\boldsymbol{x}]^\star$, the* inverse system *generated by D is the vector space spanned by the elements $p \star \delta$ for $\delta \in D$, $p \in R$.*

For $D \subset \mathbb{K}[[\boldsymbol{y}]]$, the inverse system generated by D is the vector space spanned by the elements in D and all their derivatives.

For $\omega_1, \ldots, \omega_m \in \mathbb{K}[\boldsymbol{y}]$, we denote by $\mathscr{D}(\omega_1, \ldots, \omega_m)$ the inverse system of $\omega_1, \ldots, \omega_m$, generated by ω_i and all the derivatives $\partial_{\boldsymbol{y}}^{\alpha}(\omega_i)$, $\alpha \in \mathbb{N}^n$, $i = 1, \ldots, m$. Let $\mu(\omega_1, \ldots, \omega_m)$ denote its dimension.

Lemma 2.4.2 *For $\omega \in \mathbb{K}[\boldsymbol{y}]$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{K}^n$,*

$$\mathscr{D}(\omega \mathfrak{e}_{\xi}(\boldsymbol{y})) = \mathscr{D}(\omega)\mathfrak{e}_{\xi}$$

Proof. We have, for $i = 1, \ldots, n$, $\partial_{y_i}(\omega \mathfrak{e}_{\xi}(\boldsymbol{y})) = \partial_{y_i}(\omega)\mathfrak{e}_{\xi} + \xi_i \omega \mathfrak{e}_{\xi} \in \mathscr{D}(\omega)\mathfrak{e}_{\xi}(\boldsymbol{y})$. This shows, on one hand, that $\mathscr{D}(\omega \mathfrak{e}_{\xi}(\boldsymbol{y})) \subset \mathscr{D}(\omega)\,\mathfrak{e}_{\xi}$. Since $\partial_{y_i}(\omega)\,\mathfrak{e}_{\xi} = \partial_{y_i}(\omega \mathfrak{e}_{\xi}) - \xi_i \omega\, \mathfrak{e}_{\xi}$, this shows on the other hand, that $\mathscr{D}(\omega)\mathfrak{e}_{\xi} \subset \mathscr{D}(\omega \mathfrak{e}_{\xi})$ and the equality. $\square$

Example 2.4.3 *Let $I = ((x_1 - 1)^2, (x_2 - 1)^2) \subset \mathbb{K}[x_1, x_2]$. Then,*

$$I^{\perp} = \langle \mathfrak{e}_{(1,1)}, y_1 \mathfrak{e}_{(1,1)}, y_2 \mathfrak{e}_{(1,1)}, y_1 y_2\, \mathfrak{e}_{(1,1)} \rangle = \mathscr{D}(y_1 y_2\, \mathfrak{e}_{(1,1)}) = \mathscr{D}(y_1 y_2)\mathfrak{e}_{(1,1)}.$$

and $\mu(y_1 y_2) = 4$. This is the multiplicity of the unique point $(1, 1)$ defined by I.

Chapter 3

Artinian algebra

In this section, we recall the properties of Artinian algebras. Let $I \subset \mathbb{K}[\boldsymbol{x}]$ be an ideal and let $\mathscr{A} = \mathbb{K}[\boldsymbol{x}]/I$ be the associated quotient algebra.

Definition 3.0.4 *The quotient algebra $\mathscr{A}$ is artinian if* $\dim_{\mathbb{K}}(\mathscr{A}) < \infty$.

Notice that if $\mathbb{K}$ is a subfield of a field $\mathbb{L}$, we denote by $\mathscr{A}_{\mathbb{L}} = \mathbb{L}[\boldsymbol{x}]/I_{\mathbb{L}} = \mathscr{A} \otimes \mathbb{L}$ where $I_{\mathbb{L}} = I \otimes \mathbb{L}$ is the ideal of $\mathbb{L}[\boldsymbol{x}]$ generated by the elements in I. As the dimension does not change by extension of the scalars, we have $\dim_{\mathbb{K}}(\mathbb{K}[\boldsymbol{x}]/I) = \dim_{\mathbb{L}}(\mathbb{L}[\boldsymbol{x}]/I_{\mathbb{L}}) = \dim_{\mathbb{L}}(\mathscr{A}_{\mathbb{L}})$. In particular, $\mathscr{A}$ is artinian if and only if $\mathscr{A}_{\bar{\mathbb{K}}}$ is artinian, where $\bar{\mathbb{K}}$ is the algebraic closure. For the sake of simplicity, we are going to assume hereafter that $\mathbb{K}$ *is algebraically closed*.

3.1 Univariate polynomials

Let us analyze first, the ring $R = \mathbb{K}[x]$ of univariate polynomials in the variable x and coeficient in $\mathbb{K}$. Let I be the ideal of R generated by the polynomial $f = f_d x^d + \cdots + f_0$ of degree d ($f_d \neq 0$).

The vector space $\mathscr{A} = \mathbb{K}[x]/(f)$ is of dimension d, and admit as basis $(1, x, \ldots, x^{d-1})$.

Assume that the field $\mathbb{K}$ is algebraically closed and that the roots of f are simple: $f = f_d \prod_{i=1}^{d}(x - \zeta_i)$, with $\zeta_i \neq \zeta_j$ si $i \neq j$.

Let M_x be the operator of multiplication by x in $\mathscr{A}$:

$$\begin{aligned} M_x : \mathscr{A} &\rightarrow \mathscr{A} \\ a &\mapsto a\,x. \end{aligned}$$

The matrix of M_x in the basis $(1, x, \ldots, x^{d-1})$ is the compagnon matrix

$$M_x = \begin{pmatrix} 0 & \cdots & 0 & -\frac{f_0}{f_d} \\ 1 & \ddots & \vdots & \vdots \\ & \ddots & 0 & \vdots \\ 0 & & 1 & -\frac{f_{d-1}}{f_d} \end{pmatrix}.$$

The last column of M_x corresponds of the Euclidean division of x^d by f. The characteristic polynomial of M_x is f and the eigenvalues of M_x are the roots $\zeta_1, \ldots, \zeta_d$ of f. As these eigenvalues are assumed to be distinct, the matrix M_x is diagonalisable.

Let p be an element of $\mathbb{K}[x]$. The eigenvalues of the multiplication by p in $\mathscr{A}$ are $p(\zeta_1), \ldots, p(\zeta_d)$ since they are the diagonal terms of M_p in the basis of eigenvectors of M_x. Let us describe this basis of common eigenvectors of the operators M_p, $p \in \mathbb{K}[x]$. Let

$$\boldsymbol{u}_i(x) = \prod_{j=1, j\neq i}^{d} \left(\frac{x - \zeta_j}{\zeta_i - \zeta_j} \right)$$

the i^{th} *Lagrange interpolation polynomial* of f. The elements $\boldsymbol{u}_i(\boldsymbol{u}_i - 1)$, $(x - \zeta_i)\,\boldsymbol{u}_i$, $\boldsymbol{u}_i\,\boldsymbol{u}_j$ for $j \neq i$ vanish at all the roots of f. Thus they are divisible by f and we have in $\mathscr{A}$,

$$\boldsymbol{u}_i^2 \equiv \boldsymbol{u}_i \quad , \quad x\,\boldsymbol{u}_i \equiv \zeta_i \boldsymbol{u}_i \quad , \quad \boldsymbol{u}_i \boldsymbol{u}_j \equiv 0 \;\; si \;\; i \neq j.$$

As $1 = \boldsymbol{u}_1 + \cdots + \boldsymbol{u}_d$, we have for any $a \in \mathbb{K}[x]$

$$a \equiv a(\zeta_1)\,\boldsymbol{u}_1 + \cdots + a(\zeta_d)\,\boldsymbol{u}_d \tag{3.1}$$

in $\mathscr{A}$. This implies that $\mathscr{A} \equiv \mathbb{K}\,\boldsymbol{u}_1 \oplus \cdots \oplus \mathbb{K}\,\boldsymbol{u}_d$. The family $\boldsymbol{u} = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d)$ is a basis of $\mathscr{A}$. The elements $\boldsymbol{u}_i$ are called *orthogonal idempotents* since they satisfy the relations: $\boldsymbol{u}_i^2 \equiv \boldsymbol{u}_i$, $\boldsymbol{u}_i \boldsymbol{u}_j \equiv 0$ if $i \neq j$. Moreover, as $M_x(\boldsymbol{u}_i) = x\,\boldsymbol{u}_i \equiv \zeta_i \boldsymbol{u}_i$ and $\boldsymbol{u}_i$ is an eigenvector of M_x for the eigenvalue ζ_i.

Let us consider now the dual $\mathscr{A}^\star$ of $\mathscr{A}$, that is, the vector space of linear forms on the vector space $\mathscr{A}$. It is a vector space of dimension $d = \dim \mathscr{A}$. The dual basis of the basis $(1, x, \ldots, x^{d-1})$ of $\mathscr{A}$ is denoted $\boldsymbol{d} = (\boldsymbol{d}^0, \ldots, \boldsymbol{d}^{d-1})$. The decomposition of an element $\sigma \in \mathscr{A}^\star$ in this basis is of the form

$$\sigma = \sigma(1)\boldsymbol{d}^0 + \cdots + \sigma(x^{d-1})\boldsymbol{d}^{d-1}.$$

Let $p \in \mathbb{K}[x]$ and $r = r_0 + \cdots + r_{d-1}\, x^{d-1}$ its representative in the quotient algebra $\mathscr{A}$, that is, the remainder in the Euclidean division of f by p: $p = f\,q + r$ with $\deg(r) < d$. For $\sigma \in \mathscr{A}^\star$, we have

$$\sigma(p) = \sigma(r) = r_0\,\sigma(1) + \cdots + r_{d-1}\,\sigma(x^{d-1}).$$

Among the elements of $\mathscr{A}^\star$, we have the evaluations $\mathfrak{e}_\zeta : p \mapsto p(\zeta)$ at the roots σ_i of f. The identity (3.1) shows that $\mathfrak{e} = (\mathfrak{e}_{\zeta_1}, \ldots, \mathfrak{e}_{\zeta_d})$ is the basis of $\mathscr{A}^\star$ dual to the basis $\boldsymbol{u}$ of $\mathscr{A}$.

The transpose M_x^t of the operator on multiplication M_x is by definition

$$\begin{array}{rcl} M_x^t : \mathscr{A}^\star & \rightarrow & \mathscr{A}^\star \\ \sigma & \mapsto & \sigma \circ M_x . \end{array}$$

The matrix of M_x^t in the basis $\boldsymbol{d}$ of $\mathscr{A}^\star$ is tthe transpose of the matrix of M_x in the basis $(1, x, \ldots, x^{d-1})$ of $\mathscr{A}$.

As for all $a \in \mathscr{A}$,

$$M_x^t(\mathfrak{e}_{\zeta_i}))(a) = \mathfrak{e}_{\zeta_i}(x\,a) = (\zeta_i \mathfrak{e}_{\zeta_i})(a),$$

we have $M_x^t(\mathfrak{e}_{\zeta_i}) = \zeta_i \mathfrak{e}_{\zeta_i}$ and $\mathfrak{e}_{\zeta_i}$ is an eigenvector of M_x^t for the eigenvalue ζ_i. This implies that for all $p \in \mathscr{A}$, $\mathfrak{e}_{\zeta_i}$ is an eigenvector of M_p^t for the eigenvalue $p(\zeta_i)$. The operators M_p^t for $p \in \mathbb{K}[x]$ share a family of common eigenvectors.

We are going to see that many of these properties generalize to Artinian algebra associated to polynomials in several variables.

3.2 Algebraic structure

A classical result states that the quotient algebra $\mathscr{A} = \mathbb{K}[\boldsymbol{x}]/I$ is Artinian (i.e. of finite dimensional), if and only if, $\mathscr{V}_{\bar{\mathbb{K}}}(I)$ is finite, that is, I defines a finite number of (isolated) points in $\bar{\mathbb{K}}^n$ (see e.g. [CLO92][Theorem 6] or [EM07b][Theorem 4.3]). Moreover, we have the following structure theorem (see [EM07b][Theorem 4.9]):

Theorem 3.2.1 *Let $\mathscr{A} = \mathbb{K}[\boldsymbol{x}]/I$ be an artinian algebra of dimension r defined by an ideal I. Then we have a decomposition into a direct sum of subalgebras*

$$\mathscr{A} = \mathscr{A}_{\xi_1} \oplus \cdots \oplus \mathscr{A}_{\xi_{r'}} \tag{3.2}$$

where

- *$\mathscr{V}(I) = \{\xi_1, \ldots, \xi_{r'}\} \subset \bar{\mathbb{K}}^n$ with $r' \leqslant r$.*
- *$I = Q_1 \cap \cdots \cap Q_{r'}$ is a minimal primary decomposition of I where Q_i is $\boldsymbol{m}_{\xi_i}$-primary with $\boldsymbol{m}_{\xi_i} = (x_1 - \xi_{i,1}, \ldots, x_n - \xi_{i,n})$.*
- *$\mathscr{A}_{\xi_i} \equiv \mathbb{K}[\boldsymbol{x}]/Q_i$ and $\mathscr{A}_{\xi_i} \cdot \mathscr{A}_{\xi_j} \equiv 0$ if $i \neq j$.*

We check that $\mathscr{A}$ localized at $\boldsymbol{m}_{\xi_i}$ is the local algebra $\mathscr{A}_{\xi_i}$. The dimension of $\mathscr{A}_{\xi_i}$ is the *multiplicity* of the point ξ_i in $\mathscr{V}(I)$.

The projection of 1 on the sub-algebras $\mathscr{A}_{\xi_i}$ as

$$1 \equiv \boldsymbol{u}_{\xi_1} + \cdots + \boldsymbol{u}_{\xi_{r'}}$$

with $\boldsymbol{u}_{\xi_i} \in \mathscr{A}_{\xi_i}$ yields the so-called *idempotents* $\boldsymbol{u}_{\xi_i}$ associated to the roots ξ_i. By construction, they satisfy the following relations in $\mathscr{A}$, which characterize them:

- $1 \equiv \boldsymbol{u}_{\xi_1} + \cdots + \boldsymbol{u}_{\xi_{r'}}$,
- $\boldsymbol{u}_{\xi_i}^2 \equiv \boldsymbol{u}_{\xi_i}$ for $i = 1, \ldots, r'$,
- $\boldsymbol{u}_{\xi_i} \boldsymbol{u}_{\xi_j} \equiv 0$ for $1 \leqslant i, j \leqslant r'$ and $i \neq j$.

3.3 Roots from the algebraic structure

The solutions $\mathscr{V}(I) = \{\xi_1, \ldots, \xi_{r'}\}$ can be recovered by linear algebra, from the multiplicative structure of $\mathscr{A}$, using the properties of the following operators:

Definition 3.3.1 *Let g be a polynomial in $\mathscr{A}$. The g-multiplication operator $\mathscr{M}_g$ is defined by*

$$\begin{array}{rccl} \mathscr{M}_g: & \mathscr{A} & \rightarrow & \mathscr{A} \\ & h & \mapsto & \mathscr{M}_g(h) = gh. \end{array}$$

The transpose application $\mathscr{M}_g^t$ of the g-multiplication operator $\mathscr{M}_g$ is defined by

$$\begin{array}{rccl} \mathscr{M}_g^t: & \mathscr{A}^* & \rightarrow & \mathscr{A}^* \\ & \sigma & \mapsto & \mathscr{M}_g^t(\sigma) = \sigma \circ \mathscr{M}_g = g \star \sigma. \end{array}$$

Let $B = \{b_1, \ldots, b_r\}$ be a basis in $\mathscr{A}$ and B^* its dual basis in $\mathscr{A}^*$. We denote by M_g^B (or simply M_g when there is no ambiguity on the basis) the matrix of $\mathscr{M}_g$ in the basis B. As the matrix $(M_g^B)^t$ of the transpose application $\mathscr{M}_g^t$ in the dual basis B^* in $\mathscr{A}^*$ is the transpose of the matrix M_g^B of the application $\mathscr{M}_g$ in the basis B in $\mathscr{A}$, the eigenvalues are the same for both matrices.

The main property we will use is the following (see e.g. [EM07b]):

Proposition 3.3.2 *Let I be an ideal of $R = \mathbb{K}[\boldsymbol{x}]$ and suppose that $\mathscr{V}(I) = \{\xi_1, \xi_2, \ldots, \xi_r\}$. Then*

- *for all $g \in \mathscr{A}$, the eigenvalues of $\mathscr{M}_g$ and $\mathscr{M}_g^t$ are the values $g(\xi_1), \ldots, g(\xi_r)$ of the polynomial g at the roots with multiplicities $\mu_i = \dim \mathscr{A}_{x_i}$.*
- *The eigenvectors common to all $\mathscr{M}_g^t$ with $g \in \mathscr{A}$ are - up to a scalar - the evaluations $\mathfrak{e}_{\xi_1}, \ldots, \mathfrak{e}_{\xi_r}$.*

Remark 3.3.3 *If $B = \{b_1, \ldots, b_r\}$ is a basis of $\mathscr{A}$, then the coefficient vector of the evaluation*

$$\mathbf{e}_{\xi_i} = \sum_{\beta \in \mathbb{N}^n} \xi_i^\beta \frac{\boldsymbol{y}^\beta}{\beta!} + \cdots$$

in the dual basis of $\mathscr{A}^*$ is $\left[\langle \mathfrak{e}_{\xi_i} | b_j \rangle\right]_{\beta \in B} = [b_j(\xi_i)]_{i=1\ldots r} = B(\xi_i)$. The previous proposition says that if M_g is the matrix of $\mathscr{M}_g$ in the basis B of $\mathscr{A}$, then

$$M_g^t B(\xi_i) = g(\xi_i) B(\xi_i).$$

If moreover the basis B contains the monomials $1, x_1, x_2, \ldots, x_n$, then the common eigenvectors of M_g^t are of the form $\boldsymbol{v}_i = c[1, \xi_{i,1}, \ldots, \xi_{i,n}, \ldots]$ and the root ξ_i can be computed from the coefficients of $\boldsymbol{v}_i$ by taking the ratio of the coefficients of the monomials $x_1, \ldots, x_n$ by the coefficient of 1: $\xi_{i,k} = \frac{\boldsymbol{v}_{i,k+1}}{\boldsymbol{v}_{i,1}}$. Thus computing the common eigenvectors of all the matrices M_g^t for $g \in \mathscr{A}$ yield the roots $\boldsymbol{\xi}_i$ ($i = 1, \ldots, r$). In practice, it is enough to compute the common eigenvectors of $M_{x_1}^t, \ldots, M_{x_n}^t$, since $\forall g \in \mathbb{K}[\boldsymbol{x}], M_g^t = g(M_{x_1}^t, \ldots, M_{x_n}^t)$.

3.4 The dual of an Artinian algebra

The dual $\mathscr{A}^* = \mathrm{Hom}_{\mathbb{K}}(\mathscr{A}, \mathbb{K})$ of $\mathscr{A} = \mathbb{K}[\boldsymbol{x}]/I$ is naturally identified with the sub-space

$$I^\perp = \{\sigma \in \mathbb{K}[\boldsymbol{x}]^* = \mathbb{K}[[\boldsymbol{y}]] \mid \forall p \in I, \sigma(p) = 0\}$$

of $\mathbb{K}[\boldsymbol{x}]^* = \mathbb{K}[[\boldsymbol{y}]]$ As I is stable by multiplication by the variables x_i, the orthogonal $I^\perp = \mathscr{A}^*$ is stable by the derivations $\frac{d}{\mathrm{d}y_i}$. In the case of a primary ideal, the orthogonal has a simple form [Mac16], [Ems78], [Mou96]:

Proposition 3.4.1 *Let Q be a primary ideal for the maximal ideal $\mathfrak{m}_\xi$ of the point $\xi \in \mathbb{K}^n$ and let $\mathscr{A}_\xi = \mathbb{K}[\boldsymbol{x}]/Q$. Then*

$$Q^\perp = \mathscr{A}_\xi^* = \mathscr{D}_\xi(Q) \cdot \mathfrak{e}_\xi(\boldsymbol{y}),$$

where $\mathscr{D}_\xi(Q) = \{\omega(\boldsymbol{y}) \in \mathbb{K}[\boldsymbol{y}] \mid \forall q \in Q, \omega(\partial_1, \ldots, \partial_n)(q)(\xi) = 0\}$ is the set of differential polynomials that vanish on Q at the point ξ.

The vector space $\mathscr{D}_\xi(Q) \subset \mathbb{K}[\boldsymbol{y}]$ is called the *inverse system* of Q. As Q is an ideal, $Q^\perp = \mathscr{D}_\xi(Q) \cdot \mathfrak{e}_\xi(\boldsymbol{y})$ is stable by the derivations $\frac{d}{\mathrm{d}y_i}$, and so is $\mathscr{D}_\xi(Q)$.

Lemma 3.4.2 *If $I = Q_1 \cap \cdots \cap Q_{r'}$ is a minimal primary decomposition of an ideal $I \subset \mathbb{K}[\boldsymbol{x}]$ with $\mathscr{A} = \mathbb{K}[\boldsymbol{x}]/I$ artinian and Q_i $\boldsymbol{m}_{\xi_i}$-primary, then*

$$\mathscr{A}^\star = I^\perp = Q_1^\perp \oplus \cdots \oplus Q_{r'}^\perp = \mathscr{A}_{\xi_1}^* \oplus \cdots \oplus \mathscr{A}_{\xi_{r'}}^*$$

with $\mathscr{A}_{\xi_i}^\star = \boldsymbol{u}_{\xi_i} \star \mathscr{A}^\star$.

Proof. As $I = Q_1 \cap \cdots \cap Q_{r'}$, $I^{\perp} = \sum_{i=1}^{r'} Q_i^{\perp}$. As $Q_i^{\perp} \cap Q_j^{\perp} = (Q_i + Q_j)^{\perp} = (1)^{\perp} = \{0\}$ for $i \neq j$, we have the decomposition of $\mathscr{A}^{\star}$ as a direct sum:

$$\mathscr{A}^{\star} = I^{\perp} = \bigoplus_{i=1}^{r'} Q_i^{\perp} = \bigoplus_{i=1}^{r'} \mathscr{A}_i^{\star}$$

Since $\sum_{i=1}^{r'} \boldsymbol{u}_{\xi_i} \equiv 1$ in $\mathscr{A}$, any element $\sigma \in \mathscr{A}^{*}$ decomposes as

$$\sigma = \boldsymbol{u}_{\xi_1} \star \sigma + \cdots + \boldsymbol{u}_{\xi_{r'}} \star \sigma. \tag{3.3}$$

As we have $\boldsymbol{u}_{\xi_i} \star \sigma(\mathscr{A}_{\xi_j}) = \sigma(\boldsymbol{u}_{\xi_i} \mathscr{A}_{\xi_j}) = 0$ for $i \neq j$, we deduce that $\boldsymbol{u}_{\xi_i} \star \sigma \in \mathscr{A}_{\xi_i}^{*} = Q_i^{\perp}$. The decomposition (3.3) for any $\sigma \in \mathscr{A}^{\star}$ implies that $\mathscr{A}_{\xi_i}^{\star} = \boldsymbol{u}_{\xi_i} \star \mathscr{A}^{\star}$. □

As we have

$$Q_i^{\perp} = \mathscr{D}_{\xi_i}(Q_i)\, \mathfrak{e}_{\xi_i} \subset \mathscr{D}_{\xi}(I)\, \mathfrak{e}_{\xi_i} \subset I^{\perp} \cap (\mathbb{K}[\boldsymbol{y}]\, \mathfrak{e}_{\xi_i}),$$

we deduce from the previous lemma that $Q_i^{\perp} = \mathscr{D}_{\xi_i}(I)\, \mathfrak{e}_{\xi}$ where $\mathscr{D}_{\xi_i}(I)$ is the set of differential polynomials that vanish on I at the point ξ. This can be exploited to compute efficiently the inverse system of a multiple point ξ_i from the generators of the ideal I (see e.g. [Mou96]).

From Proposition 3.4.1 and Lemma 3.4.2, we deduce the following result:

Theorem 3.4.3 *Assume that $\mathbb{K}$ is algebraically closed. Let $\mathscr{A}$ be an Artinian algebra of dimension r with $\mathscr{V}(I) = \{\xi_1, \dots, \xi_{r'}\} \subset \mathbb{K}^n$. Let $D_i = \mathscr{D}_{\xi_i}(I) \subset \mathbb{K}[\boldsymbol{y}]$ be the vector space of differential polynomials $\omega(\boldsymbol{y}) \in \mathbb{K}[\boldsymbol{y}]$ such that $\forall p \in I, \omega(\partial_1, \dots, \partial_n)(p)(\xi_i) = 0$. Then D_i is stable by the derivations $\frac{d}{\mathrm{d}y_i}$, $i = 1, \dots, n$. It is of dimension μ_i with $\sum_{i=1}^{r'} \mu_i = r$. Any elements σ of $\mathscr{A}^{*}$ has a unique decomposition of the form*

$$\sigma(\boldsymbol{y}) = \sum_{i=1}^{r'} \omega_i(\boldsymbol{y}) \mathfrak{e}_{\xi_i}(\boldsymbol{y}), \tag{3.4}$$

with $\omega_i(\boldsymbol{y}) \in D_i \subset \mathbb{K}[\boldsymbol{y}]$, which is uniquely determined by values $\langle \sigma | b_i \rangle$ for a basis $B = \{b_1, \dots, b_r\}$ of $\mathscr{A}$. Moreover, any element of this form is in $\mathscr{A}^{*}$.

Proof. For any polynomial $\omega(\boldsymbol{y}) \in \mathbb{K}[\boldsymbol{y}]$, such that $\forall \xi \in \mathscr{V}(I), \forall p \in I, \omega(\partial_1, \dots, \partial_n)(p)(\xi) = 0$ $\omega(\partial_1, \dots, \partial_n)(p)(\xi) = 0$, the element $\omega(\boldsymbol{y})\mathfrak{e}_{\xi}(\boldsymbol{y})$ is in $I^{\perp}$. Thus an element of the form (3.4) is in $I^{\perp} = \mathscr{A}^{*}$.

Let us prove that any element $\sigma \in \mathscr{A}^{*}$ is of the form (3.4). By the relation (3.3), σ decomposes as $\sigma = \sum_{i=1}^{r'} \boldsymbol{u}_{\xi_i} \star \sigma$ with $\boldsymbol{u}_{\xi_i} \star \sigma \in \mathscr{A}_{\xi_i}^{*} = Q_i^{\perp}$. By Proposition 3.4.1, $Q_i^{\perp} = D_i \mathfrak{e}_{\xi_i}(\boldsymbol{y})$, where $D_i = \mathscr{D}_{\xi_i}(Q_i)$ is the set of differential polynomials which vanish at ξ_i, on Q_i and thus on I. Thus $\boldsymbol{u}_{\xi_i} \star \sigma$ is of the form $\boldsymbol{u}_{\xi_i} \star \sigma = \omega_i(\boldsymbol{y})\mathfrak{e}_{\xi_i}(\boldsymbol{y})$ with $\omega_i(\boldsymbol{y}) \in D_i \subset \mathbb{K}[\boldsymbol{y}]$. By Lemma 2.2.4, its decomposition as a sum of polynomial exponentials $\sigma(\boldsymbol{y}) = \sum_{i=1}^{r'} \omega_i(\boldsymbol{y})\mathfrak{e}_{\xi_i}(\boldsymbol{y})$ is unique. This concludes the proof. □

Theorem 3.4.3 can be reformulated in terms of solutions of partial differential equations, using the relation between Artinian algebras and polynomial-exponentials $\mathscr{P}ol\mathscr{E}xp$. This duality between polynomials equations and partial differential equations with constant coefficients goes back to [Riq10] and has been further studied and extended for instance in [Grö], [Ems78], [Ped99], [OP01], [HT04]. In the case of a non-Artinian algebra, the solutions on an open convex domain are in the closure of the set of polynomial-exponential solutions (see e.g. [Mal56][Théorème 2] or [Hor90][Theorem 7.6.14]).

The following result gives an explicit description of the solutions of partial differential equations associated to Artinian algebras, as special elements of $\mathscr{P}ol\mathscr{E}xp$, with polynomial weights in the inverse systems of the points of the characteristic variety of the differential system:

Theorem 3.4.4 *Let $p_1,\dots,p_s \in \mathbb{C}[x_1,\dots,x_n]$ be polynomials such that $\mathbb{C}[\boldsymbol{x}]/(p_1,\dots,p_s)$ is finite dimensional over $\mathbb{C}$. Let $\Omega \subset \mathbb{R}^n$ be a convex open domain of $\mathbb{R}^n$. A function $f \in C^\infty(\Omega)$ is a solution of the system of partial differential equations*

$$p_1(\partial_1,\dots,\partial_n)(f)=0,\dots,p_s(\partial_1,\dots,\partial_n)(f)=0 \tag{3.5}$$

if and only if it is of the form

$$f(\boldsymbol{y})=\sum_{i=1}^{r}\omega_i(\boldsymbol{y})e^{\xi_i\cdot \boldsymbol{y}}$$

with $\mathscr{V}_{\mathbb{C}}(p_1,\dots,p_s)=\{\xi_1,\dots,\xi_r\}\subset\mathbb{C}^n$ and $\omega_i(\boldsymbol{y})\in D_i\subset\mathbb{C}[\boldsymbol{y}]$ where $D_i=\mathscr{D}_{\xi_i}((p_1,\dots,p_s))$ is the space of differential polynomials, which vanish on the ideal $(p_1,\dots,p_s)$ at ξ_i.

Proof. By a shift of the variables, we can assume that Ω contains 0. A solution of f of (3.5) in $C^\infty(\Omega)$ has a Taylor series expansion $f(\boldsymbol{y})\in\mathbb{C}[[\boldsymbol{y}]]$ at $0\in\Omega$, which defines an element of $\mathbb{C}[\boldsymbol{x}]^*$. By Lemma 2.2.1, f is a solution of the system (3.5) if and only if we have $p_1\star f(\boldsymbol{y})=0,\dots,p_s\star f(\boldsymbol{y})=0$. Equivalently, $f(\boldsymbol{y})\in I^\perp$ where $I=(p_1,\dots,p_s)$ is the ideal of $\mathbb{K}[\boldsymbol{x}]$ generated by $p_1,\dots,p_s$. If $\mathscr{A}=\mathbb{K}[\boldsymbol{x}]/I$ is finite dimensional, i.e. Artinian, Theorem 3.4.3 implies that the Taylor series $f(\boldsymbol{y})$ is in $I^\perp$, if and only if, it is of the form:

$$f(\boldsymbol{y})=\sum_{i=1}^{r}\omega_i(\boldsymbol{y})e^{\xi_i\cdot \boldsymbol{y}} \tag{3.6}$$

with $\mathscr{V}_{\mathbb{C}}(p_1,\dots,p_s)=\{\xi_1,\dots,\xi_r\}\subset\mathbb{C}^n$ and $\omega_i(\boldsymbol{y})\in D_i=\mathscr{D}_{\xi_i}(I)\subset\mathbb{C}[\boldsymbol{y}]$ where D_i is the space of differential polynomials which vanish on $I=(p_1,\dots,p_s)$ at ξ_i. The polynomial-exponential function (3.6) is an analytic function with an infinite radius of convergence, which is a solution of the partial differential system (3.5) on Ω. By unicity of the solution with given derivatives at $0\in\Omega$, $\sum_{i=1}^{r}\omega_i(\boldsymbol{y})e^{\xi_i\cdot\boldsymbol{y}}$ coincides with f on all the domain $\Omega\subset\mathbb{R}^n$. □

Here is another reformulation of Theorem 3.4.3 in terms of *convolution* or *cross-correlation* of sequences:

Theorem 3.4.5 *Let $p_1, \ldots, p_s \in \mathbb{C}[x_1, \ldots, x_n]$ be polynomials such that $\mathbb{C}[\boldsymbol{x}]/(p_1, \ldots, p_s)$ is finite dimensional over $\mathbb{C}$. The generating series of the sequences $\sigma = (\sigma_\alpha) \in \mathbb{C}^{\mathbb{N}^n}$ which satisfy the system of difference equations*

$$p_1 \star \sigma = 0, \ldots, p_s \star \sigma = 0 \tag{3.7}$$

are of the form

$$\sigma(\boldsymbol{y}) = \sum_{\alpha \in \mathbb{N}^n} \sigma_\alpha \frac{\boldsymbol{y}^\alpha}{\alpha!} = \sum_{i=1}^{r} \omega_i(\boldsymbol{y}) e^{\xi_i \cdot \boldsymbol{y}}$$

with $\mathcal{V}_{\mathbb{C}}(p_1, \ldots, p_s) = \{\xi_1, \ldots, \xi_r\} \subset \mathbb{C}^n$ and $\omega_i(\boldsymbol{y}) \in D_i \subset \mathbb{C}[\boldsymbol{y}]$ such that $D_i = \mathcal{D}_{\xi_i}((p_1, \ldots, p_s))$ is the space of differential polynomials, which vanish on the ideal $(p_1, \ldots, p_s)$ at ξ_i.

Proof. The sequence σ is a solution of the system (3.7) if and only if $\sigma(\boldsymbol{y}) = \sum_{\alpha \in \mathbb{N}^n} \sigma_\alpha \frac{y^\alpha}{\alpha!} \in I^\perp$ where $I = (p_1, \ldots, p_s)$ is the ideal of $\mathbb{K}[\boldsymbol{x}]$ generated by $p_1, \ldots, p_s$. We deduce the form of $\sigma(\boldsymbol{y}) \in \mathcal{P}ol\mathcal{E}xp(\boldsymbol{y})$ from Theorem 3.4.3. $\square$

3.5 Roots from the dual structure

3.5.1 Notations

Let $\mathcal{M}$ be the set of monomials in the variables $x_1, \ldots, x_n$. An element of $\mathcal{M}$ is of the form $\boldsymbol{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. Its degree is $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Let $R = \mathbb{K}[\boldsymbol{x}]$ be the ring of polynomials in the variables $x_1, \ldots, x_n$ with coefficients in a field $\mathbb{K}$. For $p = \sum_{\alpha \in A} p_\alpha \boldsymbol{x}^\alpha$ with $p_\alpha \neq 0$, A is the support of p and $\deg(p) = \max_{\alpha \in A} |\alpha|$.

For $d \in \mathbb{N}$ and $F \subset R$, let $F_{\leqslant d}$ (resp. F_d) be the set of polynomials in F of degree $\leqslant d$ (resp. d).

For $f \in R$, let $f^\top$ be the homogeneous component of f of highest degree. Similarly for a set $S \subset R$, $S^\top = \{f^\top \mid f \in S\}$.

For $F \subset R$, let $\langle F \rangle$ be the $\mathbb{K}$-vector space spanned by F. Let $F^+ = F \cup x_1 F \cup \cdots x_n F$ and $\partial F = F^+ \setminus F$. For $d \in \mathbb{N}_+$, let $F_{\leqslant d} = \{mf \mid m \in \mathcal{M}, f \in F, \deg(mf) \leqslant d\}$ and $F_d = F_{\leqslant d} \setminus F_{\leqslant d-1}$.

A set $B \subset \mathcal{M}$ is *connected to* 1 if $1 \in B$ and $\forall m \in B \setminus \{1\}$, there exists $1 \leqslant i \leqslant n$ and $m' \in B$, such that $m = x_i m'$.

For $F \subset R$ and $B \subset \mathcal{M}$, $(F|B)$ is the matrix of coefficients of the polynomials for the monomials in B.

3.5.2 Truncated normal forms

As in the introduction, let $R = \mathbb{C}[x_1, \ldots, x_n]$ be the ring of polynomials in the variables $x_1, \ldots, x_n$ with coefficients in the field $\mathbb{C}$ and take $I \subset R$ defining $\delta < \infty$ points, counting multiplicities. This is equivalent to the assumption that $\dim_{\mathbb{C}}(R/I) = \delta < \infty$. A *normal* *form*, which in [DB04] is also called an ideal projector, is a map characterized by the following properties.

Definition 3.5.1 (Normal form) *A* normal form *on R w.r.t. I is an R-map $\mathcal{N} : R \to B$ where $B \subset R$ is a vector space such that*

$$0 \longrightarrow I \longrightarrow R \xrightarrow{\mathcal{N}} B \longrightarrow 0$$

is exact and $\mathcal{N}_{|B} = \mathrm{id}_B$.

From this definition, it follows that $B \simeq R/I$ and that the algebraic structure of $\mathcal{A} = R/I$ is completely determined by $\mathcal{N}$. Since $\mathcal{N}_{|B} = \mathrm{id}_B$, we have $\mathcal{N} \circ \mathcal{N} = \mathcal{N}$ and $\mathcal{N}$ is a projector with kernel I and image B.

Example 3.5.2 (Euclidean division) *Take $R = \mathbb{K}[x_1]$, $f = f_0 + \cdots + f_d x_1^d$ with $f_i \in \mathbb{K}$ and $f_d \neq 0$. By euclidean division, for all $p \in R$ there exists a unique polynomial $r \in R$ of degree $< d$ and a unique $q \in R$, such that*

$$p = q f + r.$$

The remainder r belongs to the vector space B spanned by the monomials $1, \ldots, x_1^{d-1}$. The map $\mathcal{N}$ which associates to every polynomial $p \in R$ its remainder $r = \mathcal{N}(p)$ by euclidean division by f is such that

$$0 \to (f) \to R \xrightarrow{\mathcal{N}} B \to 0$$

is exact. In other words, we have $\ker \mathcal{N} = (f)$, $\mathrm{im}\mathcal{N} = B \sim R/(f)$.

Since we want to find a representation of R/I using numerical linear algebra techniques, we will work with linear maps $\mathcal{N}$ that can be represented by a matrix. That is, we will work with restricted or *truncated* versions of normal form maps [TMVB18].

Definition 3.5.3 (Truncated normal form) *Let $B \subset V \subset R$ with $x_i \cdot B \subset V, i = 1, \ldots, n$. A* Truncated Normal Form (TNF) *on V w.r.t. I is a linear map $\mathcal{N} : V \to B$ such that $\mathcal{N}_{|B} = \mathrm{id}_B$ and $\ker \mathcal{N} = I \cap V$. That is, $\mathcal{N}$ is a projector such that*

$$0 \longrightarrow I \cap V \longrightarrow V \xrightarrow{\mathcal{N}} B \longrightarrow 0$$

is exact.

In the case B is of finite dimension δ, let $P : B \to \mathbb{C}^\delta$ be an isomorphism defining coordinates on B. Denote $N = P \circ \mathcal{N}$. The linear map N is of the form $N : f \in V \to N(f) = (\eta_1(f), \ldots, \eta_\delta(f)) \in \mathbb{C}^\delta$ with $\eta_i \in V^* \cap I^\perp = \{\lambda \in V^* \mid \forall p \in I \cap V, \lambda(p) = 0\}$. It is given by δ linear forms, which kernel is $I \cap V$ and such that $N_{|B}$ is invertible. Conversely, a map $N : V \to \mathbb{K}^\delta$ such that $\ker N = I \cap V$ and $N_{|B} = P$ invertible, defines a truncated normal form $\mathcal{N} = N_{|B}^{-1} \circ N$. Such a map N will also be called a *truncated normal form* on V, with respect to B.

Let us give conditions under which a projector is the truncated normal form of an ideal, showing that a truncated normal form is the restriction of a normal form.

We consider $B, V \subset R$ such that $B^+ \subset V$ and $N : V \to B$ a projector such that $\ker N \subset I \cap V$. We assume that V is connected to 1.

Let us define the operator of multiplication by the variable x_i as

$$\begin{aligned} M_i : \quad & B \longrightarrow B, \\ & b \mapsto N(x_i \cdot b). \end{aligned}$$

To check that N is a truncated normal form, we use the following set of commutation polynomials:

Definition 3.5.4 *For $F \subset R$ and $B \subset R$, let $\mathscr{C}_V(F)$ be the set of polynomials in V which are of the form*

1. *$x_i f$ with $f \in F$, or*
2. *$x_i f - x_j f'$ with $f, f' \in F$, $1 \leqslant i < j \leqslant n$.*

The set $\mathscr{C}_V(F)$ is called the set of commutation polynomials of F.

The subset of $\mathscr{C}_V(F)$ satisfying condition 1 (resp. 2) is denoted $\mathscr{C}_V^1(F)$ (resp. $\mathscr{C}_V^2(F)$). Notice that $\mathscr{C}_V(F) \subset \langle F^+ \rangle$.

For $F_i \subset R$, $0 \leqslant i \leqslant d$, let

$$F_{\langle d \rangle} = \langle pf \mid p \in R_{\leqslant l}, f \in F_{d-l} \rangle$$

The next theorem describes different equivalent conditions for a normal form in degree $\leqslant d$. It summarizes results which can be deduced from results in [Mou99], [MT05a], [MT08].

Theorem 3.5.5 *Let $B, V \subset R$ such that $B^+ \subset V$, V is connected to 1, $N : V \to B$ be a projector on B along $K = \ker N$. Let $V_0 = \langle 1 \rangle$, $B_0 = \langle N(1) \rangle$, and for $l \in \mathbb{N}$, $V_{l+1} = B_l^+$, $B_{l+1} = N(V_{l+1})$ and $K_l = \ker N \cap V_l$. Then for $d \geqslant 2$ the following points are equivalent:*

1. *$(M_i \circ M_j - M_j \circ M_i)_{|B_{d-2}} = 0$ for $1 \leqslant i, j \leqslant n$;*
2. *there exists a unique truncated normal form $\tilde{N} : R_{\leqslant d} \to B_d$ such that $\tilde{N}_{|V_d} = N_{|V_d}$ and $\ker \tilde{N} = K_{\langle d \rangle}$;*
3. *$K_{d-1}^+ \cap V_d \subset K_d$;*
4. *$\mathscr{C}_{V_d}(K_{d-1}) \subset K_d$;*

Proof.
1) $\Rightarrow$ 2) : By construction, we have $V_{l+1} = B_l^+ \subset V$, $N(B_l^+) = B_{l+1}$ so that $M_i : B_l \to B_{l+1}$. Let $u = N(1)$ and define

$$\begin{array}{rcl} \tilde{N} : R_{\leqslant d} & \to & B_d \\ p & \mapsto & p(\boldsymbol{M})(u). \end{array}$$

This construction is well defined since it is independent of the order in which we compose the operators M_i since they are commuting, and $u \in B_0$ and since for $p \in R_{\leqslant d}$, we have $p(\boldsymbol{M})(u) \in B_d$.

Let us show that $\tilde{N}$ is a projection of $R_{\leqslant d}$ on B_d, which extends N and such that $\ker \tilde{N} = K_{\langle d \rangle}$.

We first prove by induction on $k \in \mathbb{N}$ that for $b \in V_k$, $\tilde{N}(b) = N(b)$. For $l = 0$, $V_0 = \langle 1 \rangle$ and $\tilde{N}(1) = u = N(1)$, which shows that the hypothesis is true for $l = 0$. Let us assume that the property is true for $0 \leqslant l \leqslant d$. Any $b \in B_{l+1}$ is of the form $b = \sum_i x_i b_i'$ with $b_i' \in B_l$. Then

$$\tilde{N}(b) = \sum_i M_i(b_i'(\boldsymbol{M})(u)) = \sum_i M_i(N(b_i')) = \sum_i N(x_i\, b_i') = N(\sum_i x_i\, b_i') = N(b)$$

by the induction hypothesis and since $N_{|B_i} = \mathrm{id}_{|B_i}$.

In the next step, we prove that $K_{\langle d \rangle} \subset \ker \tilde{N}$. For $k \in K_l$ and $p \in R_{d-l}$ with $0 \leqslant l \leqslant d$, we have

$$\tilde{N}(p\,k) \;=\; p(\boldsymbol{M}) \circ k(\boldsymbol{M})(u) = p(\boldsymbol{M})(\tilde{N}(k)) = p(\boldsymbol{M})(N(k)) = p(\boldsymbol{M})(0) = 0$$

since $\tilde{N}_{|V_l} = N_{|V_l}$.

Finally, we prove by induction that $R_{\leqslant d} = B_d \oplus K_{\langle d \rangle}$. For any $p \in R$ of degree $1 \leqslant l \leqslant d$, there exist $p_i' \in R$ of degree $l-1$ such that $p = \sum_i x_i p_i'$. We have

$$p - \tilde{N}(p) = \sum_i x_i(p_i' - \tilde{N}(p_i')) + \sum_i x_i \tilde{N}(p_i') - \tilde{N}(x_i N(p_i')) + \sum_i \tilde{N}(x_i(\tilde{N}(p_i') - p_i')).$$

By induction on the degree, we have $(p_i' - \tilde{N}(p_i')) \in K_{\langle l-1 \rangle}$. Then $\sum_i x_i(p_i' - \tilde{N}(p_i')) \in K_{\langle l \rangle}$ and $\tilde{N}(\sum_i x_i(p_i' - \tilde{N}(p_i'))) = 0$ since $K_{\langle l \rangle} \subset \ker \tilde{N}$. Moreover, $b_i = \tilde{N}(p_i') \in B_{l-1}$, thus $x_i b_i \in B_{l-1}^+ = V_l$ and $x_i b_i - \tilde{N}(x_i b_i) = x_i b_i - N(x_i b_i) \in \ker N \cap V_l = K_l$.

This shows that $p - \tilde{N}(p) \in K_{\langle l \rangle}$. As for any $p \in R_{\leqslant d}$, $p = \tilde{N}(p) + p - \tilde{N}(p)$, we deduce that $R_{\leqslant d} = B_d + K_{\langle d \rangle}$. As $K_{\langle d \rangle} \subset \ker \tilde{N}$ and $\tilde{N}_{|B_d} = \mathrm{id}$, we have

$$R_{\leqslant d} = B_d \oplus K_{\langle d \rangle},$$

with $K_{\langle d \rangle} = \ker \tilde{N}_{|R_{\leqslant d}}$.
2) $\Rightarrow$ 3) : Since $K_{d-1}^+ \cap V_d \subset K_{\langle d \rangle} \cap V_d \subset \ker \tilde{N} \cap V_d = \ker N \cap V_d = K_d$ since $\tilde{N}$ coincides with N on V_d.

$3) \Rightarrow 4)$: Clear, since $\mathscr{C}(K_{d-1}) \subset K_{d-1}^+ \cap V_d$.
$4) \Rightarrow 1)$: Let $b \in B_{d-2}$, $b_1 = x_i b \in V_{d-1}$, $b_2 = x_j b \in V_{d-1}$ with $1 \leqslant i < j \leqslant n$, $k_1 = b_1 - N(b_1) \in K_{d-1}$ and $k_2 = b_2 - N(b_2) \in K_{d-1}$. As $x_i b_2 = x_j b_1 = x_i x_j b$, we have

$$\begin{aligned}(M_i \circ M_j - M_j \circ M_i)(b) &= N(x_i N(b_2)) - N(x_j N(b_1)) \\ &= N(x_i(b_2 - k_2) - x_j(b_1 - k_1)) \\ &= N(x_j k_1 - x_i k_2).\end{aligned}$$

As $x_j k_1 - x_i k_2 = x_i N(b_2) - x_j N(b_1) \in B_{d-1}^+ = V_d$ is an element of $\mathscr{C}_{V_d}(K_{d-1})$, Hypothesis (4) implies that $N(x_j k_1 - x_i k_2) = 0$. Consequently, $(M_i \circ M_j - M_j \circ M_i)_{|B_{d-2}} = 0$. □

Theorem 3.5.6 *Let $B, V \subset R$ such that $W := B^+ \subset V$, V is connected to 1 and let $N : V \to B$ be a projector such that $K := \ker N \subset I \cap W$. Then the following points are equivalent:*

1. *$(M_i \circ M_j - M_j \circ M_i) = 0$ for $1 \leqslant i, j \leqslant n$;*
2. *there exists a unique normal form $\tilde{N} : R \to B$ such that $\tilde{N}_{|W} = N_{|W}$ and $\ker \tilde{N} = (K)$;*
3. *$K^+ \cap W \subset K$;*
4. *$\mathscr{C}_W(K) \subset K$;*

Proof. Apply Theorem 3.5.5 for all $d \in \mathbb{N}$. □

Corollary 3.5.7 *Let $B, V \subset R$ such that $W := B^+ \subset V$, V is connected to 1 and let $N : V \to B$ be a truncated normal form with respect to I and $K = \ker N \cap W$. Then there exists a unique normal form $\tilde{N}$ modulo $(K) \subset I$ such that $\tilde{N}_{|W} = N$.*

Proof. Since N is a truncated normal form, $\ker N = I \cap V$ for some ideal $I \subset R$. $\mathscr{C}_W(K) \subset I \cap W = \ker N \cap W = K$, Theorem 3.5.6(4) implies that there exists a unique normal form $\tilde{N}$ modulo (K) such that $\tilde{N}_{|W} = N$ and $\ker \mathscr{N} = (K)$. □

3.5.3 Border basis

Border basis are special types of normal forms associated to sets $\mathscr{B}$ of monomials. In some works like [KR05, KK05, KK06, CM07, Kas11], the set $\mathscr{B}$ is finite and stable by division (called an order ideal). Hereafter we consider a more general case where $\mathscr{B} = \{\boldsymbol{x}^{\beta_1}, \ldots, \boldsymbol{x}^{\beta_r}\}$ is finite and connected to $1 = \boldsymbol{x}^{\beta_1}$.

We take $V = \langle \mathscr{B}^+ \rangle$ and $B = \langle \mathscr{B} \rangle$ and a projector $N : V \to B$.

For any $\boldsymbol{x}^\alpha \in \partial \mathscr{B}$,

$$f_\alpha = \boldsymbol{x}^\alpha - N(\boldsymbol{x}^\alpha) = \boldsymbol{x}^\alpha - \sum_{\boldsymbol{x}^\beta \in \mathscr{B}} c_{\alpha,\beta} \boldsymbol{x}^\beta \tag{3.8}$$

is an element of $K = \ker N$. Conversely, a family F of polynomials f_α of form (3.8) for $\boldsymbol{x}^\alpha \in \partial\mathscr{B}$ defines a unique projector $N : V \to B$ such that $N(\boldsymbol{x}^\beta) = \boldsymbol{x}^\beta$ for $\boldsymbol{x}^\beta \in \mathscr{B}$ and $N(\boldsymbol{x}^\alpha) = \sum_{\boldsymbol{x}^\beta \in \mathscr{B}} c_{\alpha,\beta}\boldsymbol{x}^\beta$ for $\boldsymbol{x}^\alpha \in \partial\mathscr{B}$. Such a family will be called a *rewriting family* for $\mathscr{B}$.

A *border basis* for $\mathscr{B}$ is a rewriting family for $\mathscr{B}$ such that $R = \langle\mathscr{B}\rangle \oplus (F)$ and the projection on $\langle\mathscr{B}\rangle$ along (F) is a normal form.

By Theorem 3.5.6, if any of the following points is satisfied:

- $M_i \circ M_j - M_j \circ M_i = 0$ where $M_i : b \in B \mapsto N(x_i b) \in B$.
- $\langle F^+\rangle \cap V = \langle F\rangle$.
- $\forall f \in \mathscr{C}_V(F), N(f) = 0$.

then N extends to a unique normal form $\mathscr{N}$ such that $\mathscr{N}_{|V} = N$ and $\ker\mathscr{N} = (F)$. That is $R = B \oplus (F)$ and F is a border basis.

We will not assume that B is known apriori or that the projection is compatible with a monomial ordering as in [KR05, CM07], since this leads to the construction of Gröbner bases, with well-developed monomial rewriting techniques but also with numerical instability problems that we want to avoid.

For the sake of simplicity, we restrict the present article to projections compatible with the usual degree. This is not a conceptual limitation.

So far, border bases have been developed essentially for zero-dimensional ideals, except in [CM07] where the projection is compatible with a monomial ordering and thus leads to Gröbner basis computation.

The main contribution of this paper is to provide a new criteria of border basis for any projection compatible with the degree on a vector space spanned by a set B of monomials connected to 1. This criteria which applies to any ideal is based on the persistence and regularity theorems of G. Gotzmann [Got78].

We describe an algorithm, which exploits a new characterization of border basis up to a given degree, and proceeds incrementally degree by degree until the regularity criteria is satisfied. This algorithm is an extension of the algorithm in [MT05a] for zero-dimensional ideals. It is complete and has no possible case of “failure” as the algorithm for zero-dimensional ideals in [Kas11]. As a byproduct, we obtain the Hilbert polynomial of the graded part of the ideal and thus the dimension and the degree of the solution set.

Let $B \subset \mathscr{M}$ be a set of monomials connected to 1 and let $d \in \mathbb{N}$.

In this section, we assume that we are given a *projection* $\pi : \langle B^+\rangle_{\leqslant d} \to \langle B\rangle_{\leqslant d}$ (ie. satisfying $\pi \circ \pi = \pi$) which is compatible with the degree: $\forall b \in \langle B^+\rangle_{\leqslant d}$, $\deg(\pi(b)) \leqslant \deg(b)$. As $\pi_{|\langle B\rangle_{\leqslant d}}$ is the identity map, $\ker\pi$ is spanned by the elements:

$$f_\alpha = \boldsymbol{x}^\alpha - \pi(\boldsymbol{x}^\alpha), \alpha \in (\partial B)_{\leqslant d}.$$

We denote by F this generating set of polynomials of $\ker\pi$ and call it the rewriting family of $\ker\pi$.

Our objective is to characterize the projections π which are the restriction of a projection $\tilde{\pi} : R \to \langle B \rangle$ such that $I := \ker \tilde{\pi}$ is the ideal generated by $\ker \pi$. In such a case, we have $R = \langle B \rangle \oplus I$ and $\tilde{\pi}$ is a normal form modulo the ideal I.

The main idea behing border basis techniques is to relate this normal form property to commutation properties of multiplication operators [Mou99]. We define the operator of multiplication by x_i associated to π as:

$$\begin{array}{rcl} M_i : \langle B \rangle_{\leqslant d-1} & \to & \langle B \rangle_{\leqslant d} \\ b & \mapsto & \pi(x_i b). \end{array}$$

As π is compatible with the degree, the image by M_i of an element of degree $\leqslant k$ is of degree $\leqslant k+1$ for $0 \leqslant k < d$.

For a monomial $\boldsymbol{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathscr{M}$ of degree $\leqslant d$, we define $\boldsymbol{x}^\alpha(\boldsymbol{M}) := M_1^{\alpha_1} \circ \cdots \circ M_n^{\alpha_n}$. It is an operator from $\langle B \rangle_{\leqslant d-|\alpha|}$ to $\langle B \rangle_{\leqslant d}$. We extend this construction by linearity and for any $p \in R_{\leqslant d}$, we define

$$p(\boldsymbol{M}) : \langle B \rangle_{\leqslant d-\deg(p)} \to \langle B \rangle_{\leqslant d}.$$

Remark 3.5.8 *As a border basis in degree $\leqslant d$ is a border basis in degree $\leqslant k$ for $0 \leqslant k \leqslant d$, this theorem implies that the restriction of $\tilde{\pi}$ to $R_{\leqslant k}$ is the projection onto $\langle B \rangle_{\leqslant k}$ along $F_{\langle \leqslant k \rangle}$.*

Remark 3.5.9 *We can define the projection $\tilde{\pi} : R_{\leqslant d} \to \langle B \rangle_{\leqslant d}$ such that for any $\boldsymbol{x}^\alpha \in \mathscr{M}_{\leqslant d}$, $\tilde{\pi}(\boldsymbol{x}^\alpha) = \boldsymbol{M}^\alpha(1) \in \langle B \rangle_{\leqslant d}$ and we extend it by linearity on $R_{\leqslant d}$. Any order in the composition of the operators M_i can be used to define a projection $\tilde{\pi}$ on a specific monomial of degree $\leqslant d$. For any of these choices, we have a projection such that $\forall p \in R_{\leqslant d}, p - \tilde{\pi}(p) \in F_{\langle \leqslant d \rangle}$.*

However, if the operators M_i commute in degree $\leqslant d-2$, then this projection $\tilde{\pi}$ is uniquely defined.

3.5.4 Characterization of TNFs

Given a linear map $N : V \to \mathbb{C}^\delta$ with $V \subset R$ a finite dimensional subvector space, what are the conditions on N, V such that N covers a TNF $\mathscr{N}$ w.r.t I?

Also, which subspaces $B \subset V$ such that $x_i \cdot B \in V, i = 1, \ldots, n$ can we identify with R/I? That is, the map $N : V \to \mathbb{C}^\delta$ might cover different TNFs $\mathscr{N} : V \to B$ and $\mathscr{N}' : V \to B'$. Theorem 3.5.10 gives an answer to these questions.

We consider a 0-dimensional ideal $I = (f_1, \ldots, f_s) \subset R$ generated by s polynomials in n variables with $\delta < \infty$ solutions in $\mathbb{C}^n$, counting multiplicities. For any ideal $J \subset R$ and $p \in R$, we denote $(J : p) = \{q \in R \mid pq \in J\}$ and $(J : p^*) = \{q \in R \mid \exists k \in \mathbb{N}\ s.t.\ p^k q \in J\}$.

Theorem 3.5.10 *Let $V \subset R$ be a finite dimensional subvector space and let $W = \{f \in V : x_i f \in V, i = 1, \ldots, n\}$. Suppose we have a $\mathbb{C}$-linear map $N : V \to \mathbb{C}^\delta$ such that*

1. *$\exists u \in V$ such that $u + I$ is a unit in R/I,*

2. $\ker(N) \subset I \cap V$,

3. $N_{|W}$ *is onto* $\mathbb{C}^{\delta}$.

Then for any δ*-dimensional vector subspace* $B \subset W$ *such that* $N_{|B}$ *is invertible we have:*

(i) *there is an isomorphism of R-modules* $B \simeq R/I$,

(ii) $V = B \oplus (I \cap V)$ *and* $I = (\langle \ker(N) \rangle : u)$,

(iii) *the maps* N_i *given by*

$$\begin{aligned} N_i: \quad B &\longrightarrow \mathbb{C}^{\delta}, \\ b &\longrightarrow N(x_i \cdot b) \end{aligned}$$

for $i = 1, \ldots, n$ *can be decomposed as* $N_i = N_{|B} \circ m_{x_i}$ *where* $m_{x_i} : B \to B$ *define the multiplications by* x_i *in* B *modulo* I *and are commuting* ($m_{x_i} \circ m_{x_j} = m_{x_j} \circ m_{x_i}$ *for* $1 \leqslant i < j \leqslant n$).

Proof.

(i) It follows from the fact that $N_{|B}$ is invertible that $V = B \oplus \ker(N)$. Let $\pi : V \to B$ be the projection onto B along $\ker(N)$ and define

$$\begin{aligned} m_{x_i}: \quad B &\longrightarrow B, \\ b &\longrightarrow \pi(x_i \cdot b). \end{aligned}$$

Then $\forall b \in B$,

$$\begin{aligned} m_{x_i}(b) &= x_i \cdot b \mod \ker(N) && (3.9) \\ &= x_i \cdot b \mod I && (3.10) \end{aligned}$$

where the last equality follows from $\ker(N) \subset I \cap V$.

For $\alpha \in \mathbb{N}^n$, we write $\boldsymbol{m}^{\alpha} = m_{x_1}^{\alpha_1} \circ \cdots \circ m_{x_n}^{\alpha_n}$ and for $f = \sum_{i=1}^{p} c_i x^{\alpha_i} \in R$ we define

$$f(\boldsymbol{m}) = \sum_{i=1}^{p} c_i \boldsymbol{m}^{\alpha_i} : B \to B.$$

Replacing u by $\pi(u)$ which is also invertible in R/I, we can assume that $u \in B$. We will show that the sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & R & \xrightarrow{\ \phi\ } & B & \longrightarrow & 0 \\ & & & & f & \longrightarrow & f(\boldsymbol{m})(u) & & \end{array}$$

with $J = \ker(\phi)$ is exact. From (3.10), we deduce that $\forall f \in R, \phi(f) = f\,u \mod I$ so that $J = \ker\phi \subset I$. If $\pi_I : R \to R/I$ is the map that sends f to its residue class in R/I, we have $\pi_I(\phi(f)) = \pi_I(f\,u)$. Hence $\pi_I(\phi(R)) = \pi_I(Ru) = R/I$ since u is invertible in R/I and $\dim_{\mathbb{C}}(\phi(R)) \geq \dim_{\mathbb{C}}(R/I) = \delta$. But also $\phi(R) \subset B$ means $\dim_{\mathbb{C}}(\phi(R)) \leq \dim_{\mathbb{C}}(B) = \delta$. We deduce that ϕ is surjective and $\pi_I : B \to R/I$ is an isomorphism. It follows that the induced map $\overline{\phi} : R/J \to B \simeq R/I$ is an isomorphism of $\mathbb{C}$-vector spaces, which implies $J = I$ since $J \subset I$. We conclude that $\overline{\phi}$ is an isomorphism of R-modules between R/I and B and its inverse is $u^{-1} \cdot \pi_I$. This proves the first point.

(ii) Moreover, $B \cap I = \{0\}$ since $\pi_I : B \to R/I$ is an isomorphism; As B is supplementary to $\ker(N)$ in V and $\ker(N) \subset I \cap V$ by hypothesis, we deduce that $I \cap V = \ker(N)$. It follows that $V = B \oplus \ker(N) = B \oplus (I \cap V)$. We have $\ker(N) \subset I$ and thus $\langle \ker(N) \rangle \subset I$. Therefore $(\langle \ker(N) : u) \subset (I : u) = I$ since u is a unit in R/I. To prove the reverse inclusion, notice that if $f \in I = J = \ker\phi$ then by the relation (3.9), $f\,u \in \langle \ker(N) \rangle$. This implies that

$$I \subset (\langle \ker(N) \rangle : u) \subset I,$$

which proves the second point.

(iii) From Equation (3.10) and the isomorphism $\overline{\phi}$ between R/I and B, we deduce that the operators m_{x_i} correspond to the multiplications by the variables x_i in the quotient algebra R/I. Thus they are commuting. By construction, we have $N_i(b) = N(x_i \cdot b) = N(\pi(x_i \cdot b)) = (N_{|B} \circ m_{x_i})(b)$, where the second equality follows from $\ker(\pi) = \ker(N)$. This concludes the proof of the third point.

□

Corollary 3.5.11 *A linear map $N : V \to \mathbb{C}^\delta$ covers a TNF $\mathcal{N}$ with respect to I if and only if N, V satisfy the conditions of Theorem 3.5.10.*

Proof. For the if direction, take any $B \subset W$ for which $N_{|B}$ is invertible and $(N_{|B})^{-1} \circ N$ is a TNF by Theorem 3.5.10. For the other implication, if N covers a TNF, then $N = P \circ \mathcal{N}$ for some isomorphism $P : B \to \mathbb{C}^\delta$, $B \subset W$. Hence $N_{|B} = P$ and $N_{|W}$ is onto $\mathbb{C}^\delta$. It is clear from the properties of TNFs that $\ker(N) = I \cap V$. For the first condition, if the isomorphism $R/I \simeq B$ is given by $\overline{\phi}$, we can take $u = \overline{\phi}(1 + I) \in B \subset V$ and we're done. □

It follows from Theorem 3.5.10 that once we have a matrix representation of $N, N_{|B}$ and the $N_i, i = 1, \ldots, n$, the matrices m_{x_i} are given by $(N_{|B})^{-1} N_i$. The eigenvalues $z_{ji}, j = 1, \ldots, \delta$ of the m_{x_i} can be computed as the generalized eigenvalues of $N_i v = \lambda N_{|B} v$. As detailed in Section 3.3, computing the eigenvalues and eigenvectors of the operators of multiplication yields the solution of the polynomial equations.

When $u = 1 \in V$, then $\forall b \in B, \phi(b) = b \mod I$. Since $B \cap I = \{0\}$, we have $\forall b \in B$, $\phi(b) = b$ and ϕ is the normal form or ideal projector on B along its kernel I. Moreover, (iii) implies that $\langle \ker(N) \rangle = I$.

By the normal form characterization proved in [Mou99, MT05b], if the set B is connected to 1 ($1 \in B$ and there exists vector spaces $B_l \subset R$ such that $B_0 = \langle()1\rangle = \mathbb{C} \subset B_1 \subset \cdots \subset B_k = B$ with $B_{l+1} \subset B_l^+$ where $B_l^+ = B_l + x_1 B_l + \cdots + x_n B_l$), then the commutation property (point (iv)) implies that $B \simeq R/I$ (point (ii)).

3.5.5 Constructing truncated normal forms

In some interesting cases, a map $N : V \to \mathbb{C}^\delta$ covering a TNF can be computed as the cokernel of a *resultant map*. Such a map is defined as follows.

Definition 3.5.12 (Resultant map) *Let $\boldsymbol{f} = [f_1, \ldots, f_s] \in R^s$. A* resultant map *w.r.t. $\boldsymbol{f}$ is a map*

$$M : \quad V_1 \times \cdots \times V_s \longrightarrow \quad V : (q_1, \ldots, q_s) \longmapsto \quad q_1 f_1 + \cdots + q_s f_s.$$

with $V_i, V \subset R$ finite dimensional vector subspaces.

Note that all resultant maps with respect to $\boldsymbol{f}$ share the property that $im(M) \subset I \cap V$ where $I = (f_1, \ldots, f_s)$. Hence, if $N = coker(M)$, we have $\ker(N) \subset I \cap V$. In the following sections, we show how TNFs are covered by the cokernel of a specific resultant map in the affine, toric, homogeneous and multihomogenous setting when I is a complete intersection.

We now show how the cokernel of a particular resultant map gives a map N and a subspace V satisfying the conditions of Theorem 3.5.10. Consider a zero-dimensional ideal $I = \langle f_1, \ldots, f_n \rangle \subset R$ such that the f_i define a system of polynomial equations that has no solutions at infinity. That is, denoting $\deg(f_i) = d_i$, we assume that the f_i are generic in the sense that there are $\delta = \prod_{i=1}^n d_i$ solutions, counting multiplicities, in $\mathbb{C}^n$. We denote these solutions by $\mathscr{V}(I) = \{z_1, \ldots, z_{\delta_0}\} \subset \mathbb{C}^n$, where $\delta_0 \leq \delta$ is the number of distinct solutions. Next, we consider a generic linear polynomial f_0. We use the classical Macaulay resultant matrix construction defined as follows. Let $\rho = \sum_{i=1}^n d_i - n + 1$, let $V = R_{\leq \rho}$ be the space of polynomials of degree $\leq \rho$ and $V_i = R_{\leq \rho - d_i}$. The associated resultant map is

$$\begin{aligned} M_0 : \quad V_0 \times V_1 \times \cdots \times V_n \quad &\longrightarrow \quad V \\ (q_0, q_1, \ldots, q_n) \quad &\longmapsto \quad q_0 f_0 + q_1 f_1 + \cdots + q_n f_n. \end{aligned}$$

There is a square submatrix M' of the matrix of M_0 such that $\det(M')$ is a nontrivial multiple of the resultant $\mathrm{Res}(f_0, f_1, \ldots, f_n)$ [CLO97, Mac02]. The monomial multiples of f_0 involved in M' have exponents in $\Sigma_0 = \{\alpha \in \mathbb{N}^n : \alpha_i < d_i, i = 1, \ldots, n\}$. The set $\mathscr{B}_0$ of monomials with exponents in Σ_0 corresponds generically to a basis (the so-called Macaulay basis) of R/I: $B_0 = \langle \mathscr{B}_0 \rangle \simeq R/I$. The matrix M' decomposes as

$$M' = \begin{bmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{bmatrix}$$

where the rows and columns of the first block M_{00} are indexed by $\mathcal{B}_0$. The matrix $\tilde{M} = \begin{bmatrix} M_{01} \\ M_{11} \end{bmatrix}$ representing monomial multiples of $f_1, \ldots, f_n$ is such that $im(\tilde{M}) \subset I \cap V$. Since for generic systems $f_1, \ldots, f_n$, the matrix M_{11} is invertible (see [Mac02], [CLO97, Chapter 3]), the rank of $\tilde{M}$ is $\dim V - \delta$. Let N be the coefficient matrix of a basis of the left null-space of $\tilde{M}$ so that $N\,\tilde{M} = 0$. Then N corresponds to a linear map $V \to \mathbb{C}^\delta$ of rank δ such that its kernel is $im(\tilde{M}) \subset I$. In fact, denoting $M = (M_0)_{|V_1 \times \cdots \times V_n}$ (i.e. $M(q_1, \ldots, q_n) = q_1 f_1 + \ldots + q_n f_n$) it satisfies

$$\ker(N) = im(\tilde{M}) = im(M) = I \cap V = I_{\leq \rho},$$

since $B_0 \cap I = \{0\}$ and M_{11} is invertible, so that any element in $im(M)$ can be projected in $B_0 \cap I$ along $im(\tilde{M})$ (i.e. $im(M) \subset im(\tilde{M}) \subset im(M)$).

In order to apply Theorem 3.5.10, we need to restrict N to a subset $W \subset V$, such that $x_i \cdot W \subset V$ and $N_{|W}$ is surjective. Let us take $W = R_{\leq \rho - 1}$. Since M_{11} is invertible, N is equivalent to the matrix $\begin{bmatrix} \mathrm{id} & -M_{01} M_{11}^{-1} \end{bmatrix}$ where the columns of the $\delta \times \delta$ identity block are indexed by the monomials in $\mathcal{B}_0$. Since $B_0 \subset W$, we deduce that $N_{|W}$ is surjective.

This leads to Algorithm 3.5.1 for computing the algebra structure of R/I. Note that in step 5 of the algorithm we make a choice of monomial basis for R/I. In order to have accurate multiplication matrices, $N_{|B}$ should be 'as invertible as possible'. A good choice here is to use QR with optimal column pivoting on the matrix $N_{|W}$, such that $\mathcal{B}$ corresponds to a well-conditioned submatrix. We use M instead of $\tilde{M}$ for numerical reasons. It leads to a more accurate computation of the null space.

Algorithm 3.5.1: Computes the structure of the algebra R/I (affine, dense case)

1: **procedure** ALGEBRASTRUCTURE($f_1, \ldots, f_n$)
2: $M \leftarrow$ the resultant map on $V_1 \times \cdots \times V_n$
3: $N \leftarrow \mathrm{null}(M^\top)^\top$
4: $N_{|W} \leftarrow$ columns of N corresponding to monomials of degree $< \rho$
5: $N_{|B} \leftarrow$ columns of $N_{|W}$ corresponding to an invertible submatrix
6: $\mathcal{B} \leftarrow$ monomials corresponding to the columns of $N_{|B}$ **for** $i = 1, \ldots, n$ **do**
7:
end
$N_i \leftarrow$ columns of N corresponding to $x_i \cdot \mathcal{B}$
8: $m_{x_i} \leftarrow (N_{|B})^{-1} N_i$
9:
10: **return** $m_{x_1}, \ldots, m_{x_n}$
11: **end procedure**

Example 3.5.13 *Consider the ideal $I = \langle f_1, f_2 \rangle \subset \mathbb{C}[x_1, x_2]$ given by*

$$\begin{aligned} f_1 &= 7 + 3x_1 - 6x_2 - 4x_1^2 + 2x_1x_2 + 5x_2^2, \\ f_2 &= -1 - 3x_1 + 14x_2 - 2x_1^2 + 2x_1x_2 - 3x_2^2. \end{aligned}$$

As illustrated in Figure 3.1, the solutions are $z_1 = (-2,3), z_2 = (3,2), z_3 = (2,1), z_4 = (-1,0)$. *The dense Macaulay matrix* M *of degree* $\rho = d_1 + d_2 - n + 1 = 3$ *is*

$$
M^\top = \begin{array}{c} \\ f_1 \\ x_1f_1 \\ x_2f_1 \\ f_2 \\ x_1f_2 \\ x_2f_2 \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 & x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 \end{array} \\
\begin{bmatrix}
7 & 3 & -6 & -4 & 2 & 5 & & & & \\
 & 7 & & 3 & -6 & & -4 & 2 & 5 & \\
 & & 7 & & 3 & -6 & & -4 & 2 & 5 \\
-1 & -3 & 14 & -2 & 2 & -3 & & & & \\
 & -1 & & -3 & 14 & & -2 & 2 & -3 & \\
 & & -1 & & -3 & 14 & & -2 & 2 & -3
\end{bmatrix}
\end{array}.
$$

Since all solutions are simple, a basis for the left null space of M *is given by* $v^{(3)}(z_i), i = 1, \dots, 4$, *where*

$$
v^{(3)}(x_1,x_2) = \begin{bmatrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 & x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 \end{bmatrix}.
$$

These are the linear functionals $\eta_i, i = 1, \dots, 4$ *in* $V^* \cap I^\perp$ *representing 'evaluation in* z_i'. *We find*

$$
N = \begin{array}{c} \\ v^{(3)}(-2,3) \\ v^{(3)}(3,2) \\ v^{(3)}(2,1) \\ v^{(3)}(-1,0) \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 & x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 \end{array} \\
\begin{bmatrix}
1 & -2 & 3 & 4 & -6 & 9 & -8 & 12 & -18 & 27 \\
1 & 3 & 2 & 9 & 6 & 4 & 27 & 18 & 12 & 8 \\
1 & 2 & 1 & 4 & 2 & 1 & 8 & 4 & 2 & 1 \\
1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0
\end{bmatrix}
\end{array}.
$$

For $\mathscr{B} = \{x_1, x_2, x_1^2, x_1x_2\}$, *the submatrices we need are*

$$
N_{|B} = \begin{bmatrix} -2 & 3 & 4 & -6 \\ 3 & 2 & 9 & 6 \\ 2 & 1 & 4 & 2 \\ -1 & 0 & 1 & 0 \end{bmatrix}, \quad
N_1 = \begin{bmatrix} 4 & -6 & -8 & 12 \\ 9 & 6 & 27 & 18 \\ 4 & 2 & 8 & 4 \\ 1 & 0 & -1 & 0 \end{bmatrix}, \quad
N_2 = \begin{bmatrix} -6 & 9 & 12 & -18 \\ 6 & 4 & 18 & 12 \\ 2 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix},
$$

corresponding to $\mathscr{B}, x_1 \cdot \mathscr{B}$ *and* $x_2 \cdot \mathscr{B}$ *respectively. The vector space* B *in this example is the space of polynomials supported in* $\mathscr{B}$. *One can check that* $N_{|B}$ *is invertible. Using Matlab, we find the eigenvalues of* $N_2 v = \lambda N_{|B} v$ *via the command* `eig`. *The eigenvalues are* $0, 1, 2, 3$ *as expected. Of course, in practice we do not know the solutions and we cannot construct the nullspace in this way. Any basis will do, since using another basis comes down to left multiplying* N *and the* N_i *by an invertible matrix. Note that* $\mathscr{B}$ *does not correspond to any monomial order and it is not connected to one, so it does not correspond to a Groebner or a border basis.*

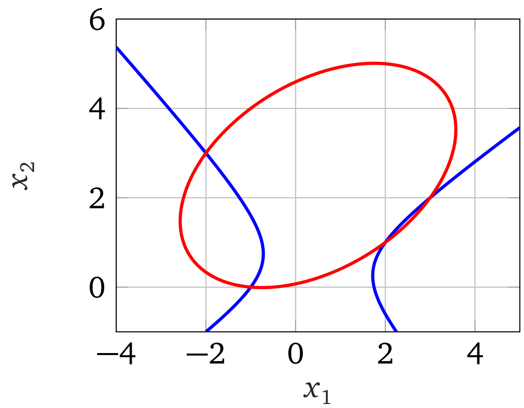


Figure 3.1: Picture in $\mathbb{R}^2$ of the algebraic curves $\mathscr{V}(f_1)$ (——) and $\mathscr{V}(f_2)$ (——) from Example 3.5.13.

Chapter 4

Decomposition from moments

In this chapter, we use the algebraic tools and the properties of Artinian algebras to recover the decomposition of moment series as polynomial-exponential series.

4.1 Hankel operators

The external product $\star$ allows us to define a Hankel operator as a multiplication operator by dual elements $\in R^{\star}$ where $R = \mathbb{K}[\boldsymbol{x}]$:

Definition 4.1.1 *The Hankel operator associated to $\sigma = (\sigma_1, \ldots, \sigma_m) \in (R^{\star})^m$ is*

$$\begin{array}{rcl} H_{\sigma} : R & \to & (R^{\star})^m \\ p & \mapsto & (p \star \sigma_1, \ldots, p \star \sigma_m) \end{array}$$

Its kernel is denoted $I_{\sigma} = \ker H_{\sigma}$. The element $\sigma \in (R^{\star})^m$ is called the symbol *of H_{σ}.*

Hereafter, we will also denote $p \star \sigma = (p \star \sigma_1, \ldots, p \star \sigma_m) \in (R^{\star})^m$ and $\langle \sigma \mid p \rangle = (\langle \sigma_1 \mid p \rangle, \ldots, \langle \sigma_m \mid p \rangle) \in \mathbb{K}^m$.

As $\forall p, q \in R,\ pq \star \sigma = p \star (q \star \sigma)$, we easily check that $I_{\sigma} = \ker H_{\sigma}$ is an *ideal* of R and that $\mathscr{A}_{\sigma} = R/I_{\sigma}$ is an algebra.

Definition 4.1.2 *The rank of an element $\sigma \in (R^\star)^m$ is the rank r of the Hankel operator H_σ.*

If the rank r of σ is finite, then the quotient $\mathcal{A}_\sigma$ of R by the kernel I_σ is of dimension $r < \infty$, i.e. $\dim \mathcal{A}_\sigma = \operatorname{rank} H_\sigma = \operatorname{rank} \sigma$ and $\mathcal{A}_\sigma$ is Artinian.

Since $\forall p(\boldsymbol{x}), q(\boldsymbol{x}) \in \mathbb{K}[\boldsymbol{x}]$, $\langle p(\boldsymbol{x}) + I_\sigma, q(\boldsymbol{x}) + I_\sigma \rangle_\sigma = \langle p(\boldsymbol{x}), q(\boldsymbol{x}) \rangle_\sigma$, we see that $\langle \cdot, \cdot \rangle_\sigma$ induces an inner product on $\mathcal{A}_\sigma$.

Definition 4.1.3 *The variety $\mathcal{V}_{\overline{\mathbb{K}}}(I_\sigma)$ is called the* characteristic variety *of σ.*

The Hankel operator can be interpreted as an operator on sequences:

$$\begin{aligned} H_\sigma : \ell_0(\mathbb{K}^{\mathbb{N}^n}) &\rightarrow (\mathbb{K}^{\mathbb{N}^n})^m \\ p = (p_\beta)_{\beta \in B \subset \mathbb{N}^n} &\mapsto \left(\left(\sum_{\beta \in B} p_\beta \sigma^1_{\alpha+\beta} \right)_{\alpha \in \mathbb{N}^n}, \dots, \left(\sum_{\beta \in B} p_\beta \sigma^m_{\alpha+\beta} \right)_{\alpha \in \mathbb{N}^n} \right) \end{aligned}$$

where $\ell_0(\mathbb{K}^{\mathbb{N}^n})$ is the set of sequences $\in \mathbb{K}^{\mathbb{N}^n}$ with a finite support and $(\sigma^i_\alpha)_{\alpha \in \mathbb{N}^n}$ is the sequence in $\mathbb{K}^{\mathbb{N}^n}$ associated to the element $\sigma \in R^\star$ (see chap. 2). This definition applies for a field $\mathbb{K}$ of any characteristic.

Given sequences $\sigma = (\sigma_1, \dots, \sigma_m)$ with $\sigma_i = (\sigma^i_\alpha)_{\alpha \in \mathbb{N}^n} \in \mathbb{K}^{\mathbb{N}^n}$ for $i = 1, \dots, m$, the kernel of H_σ is the set of polynomials $p = \sum_{\beta \in B} p_\beta \boldsymbol{x}^\beta$ such that $p = \sum_{\beta \in B} p_\beta \sigma^i_{\alpha+\beta}$ for all $\alpha \in \mathbb{N}^n$ and $i = 1, \dots, m$. This kernel is also called the set of (simultaneous) *linear recurrence relations* of the sequences $(\sigma^i_\alpha)_{\alpha \in \mathbb{N}^n}$, $i = 1, \dots, m$.

The operator H_σ can also be interpreted, via the Z-transform of the sequence $p \star \sigma$ (see Section 2.3), as the following operators on series:

$$\begin{aligned} H_\sigma : \mathbb{K}[\boldsymbol{x}] &\rightarrow \mathbb{K}[[\boldsymbol{z}]]^m \\ p = \sum_{\beta \in B} p_\beta \boldsymbol{x}^\beta &\mapsto \left(\sum_{\alpha \in \mathbb{N}^n} \left(\sum_{\beta \in B} p_\beta \sigma^i_{\alpha+\beta} \right) \boldsymbol{z}^\alpha, \dots, \sum_{\alpha \in \mathbb{N}^n} \left(\sum_{\beta \in B} p_\beta \sigma^m_{\alpha+\beta} \right) \boldsymbol{z}^\alpha \right). \end{aligned}$$

In the terms of series in $\boldsymbol{y}$, the operator H_σ is operating as follows:

$$\begin{aligned} H_\sigma : \mathbb{K}[\boldsymbol{x}] &\rightarrow \mathbb{K}[[\boldsymbol{y}]]^m \\ p &\mapsto \left(p(\partial_y)(\sigma_1), \dots, p(\partial_y)(\sigma_m) \right). \end{aligned}$$

Its kernel is spanned by the differential polynomials $p(\partial_y)$, which cancel simultaneously $\sigma_1, \dots, \sigma_m$.

Example 4.1.4 *If $\sigma = \mathfrak{e}_\xi \in R^\star$ is the evaluation at a point $\xi \in \mathbb{K}^n$, then $H_{\mathfrak{e}_\xi} : p \in R \mapsto p(\xi)\mathfrak{e}_\xi(\boldsymbol{y}) \in R^\star$. We easily check that* $\operatorname{rank} H_{\mathfrak{e}_\xi} = 1$ *since the image of $H_{\mathfrak{e}_\xi}$ is spanned by $\mathfrak{e}_\xi(\boldsymbol{y})$ and that $I_{\mathfrak{e}_\xi} = (x_1 - \xi_1, \dots, x_n - \xi_n)$.*

Example 4.1.5 *If $\sigma = (\sigma_1, \ldots, \sigma_m)$ with $\sigma_i = \sum_{i=1}^{r_i} \omega_i(\mathbf{y})\mathfrak{e}_{\xi_i}(\mathbf{y})$ then, by Lemma 2.2.1, the kernel I_σ is the set of polynomials $p \in \mathbb{K}[\mathbf{x}]$ such that $\forall q \in \mathbb{K}[\mathbf{x}]$, p is a solution of the following system of partial differential equations:*

$$\sum_{i=1}^{r_i} \omega_{k,i}(\partial)(pq)(\xi_i) = 0, k = 1, \ldots, m.$$

4.1.1 Truncated Hankel operators

In the sparse reconstruction problem, we are dealing with truncated series with known coefficients σ_α for α in a subset $\boldsymbol{a}$ of $\mathbb{N}^n$. This leads to the definition of truncated Hankel operators.

Definition 4.1.6 *For vector spaces $V \subset R$, $W = (W_1, \ldots, W_m) \subset R^m$ and $\sigma = (\sigma_1, \ldots, \sigma_m)$ with $\sigma_i \in \langle V \cdot W_i \rangle^\star$ where $V \cdot W_i = \langle v \cdot w \mid v \in V, w \in W_i \rangle \subset R$ we denote by $H_\sigma^{V,W}$ the following map:*

$$\begin{aligned} H_\sigma^{V,W} : V &\rightarrow W^* = \prod_{i=1}^m W_i^\star \\ p &\mapsto \left((p \star \sigma_1)_{|W_1}, \ldots, (p \star \sigma_m)_{|W_m}\right) \end{aligned}$$

It is called the truncated Hankel operator *on* (V, W).

When $m = 1$, $\sigma \in R^\star$ and $W = V$, the truncated Hankel operator is also denoted H_σ^V. When V (resp. W_i) is the vector space of polynomials of degree $\leqslant d \in \mathbb{N}$ (resp. $\leqslant d_i' \in \mathbb{N}$), the truncated operator is denoted $H_\sigma^{d,\mathbf{d}'}$ where $\mathbf{d}' = (d_1', \ldots, d_m')$.

If $B = \{b_1, \ldots, b_r\}$ (resp. $C^i = \{c_1^i, \ldots, c_r^i\}$) is a basis of V (resp. W_i), then the matrix of the operator $H_\sigma^{V,W}$ in B and the dual basis of $C = (C^1, \ldots, C^m)$ has a block structure of the form

$$[H_\sigma^{B,C}] = \left[\begin{array}{c} \langle \sigma_1 | b_1 c_1^1 \rangle \cdots \langle \sigma_1 | b_r c_1^1 \rangle \\ \vdots \qquad\qquad \vdots \\ \langle \sigma_1 | b_1 c_{r_1}^1 \rangle \cdots \langle \sigma_1 | b_r c_{r_1}^1 \rangle \\ \hline \vdots \\ \hline \langle \sigma_1 | b_1 c_i^m \rangle \cdots \langle \sigma_1 | b_r c_1^m \rangle \\ \vdots \qquad\qquad \vdots \\ \langle \sigma_1 | b_1 c_{r_m}^m \rangle \cdots \langle \sigma_1 | b_r c_{r_m}^m \rangle \end{array}\right].$$

Hereafter, we will use the notation $[H_\sigma^{B,C}] = [\langle \sigma_k \mid b_j c_i^k \rangle]_{c_i^k \in C, b_j \in B}$.

Example 4.1.7 *Let $\sigma = (\mathfrak{e}_{(1,0)}, \mathfrak{e}_{(1,2)}) \in (\mathbb{K}[x_1, x_2]^\star)^*$, $B = [1, x_1, x_2, x_1^2, x_1x_2, x_2^2]$, $C = (C_1, C_2) = ([1, x_1, x_2], [1, x_1, x_2])$. Then $H_\sigma^{B,C}$ is composed of two blocks, which (i, j) entry* *is obtained by evaluating σ_k on the product of the i^{th} monomial of C_k and the j^{th} monomials in B for $k = 1, 2$:*

$$H_\sigma^{B,C} = \left[\begin{array}{cccccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 2 & 1 & 2 & 4 \\ 2 & 2 & 4 & 2 & 4 & 8 \\ 4 & 4 & 8 & 4 & 8 & 16 \end{array}\right].$$

It is a matrix of rank 2 composed of two blocks of rank 1 (which rows are multiples of the first row of the block).

If $m = 1$, $\sigma \in \mathbb{K}[\boldsymbol{x}]^\star$, $B = \{\boldsymbol{x}^\beta\}_{\beta \in \boldsymbol{b}}$ and $C = \{\boldsymbol{x}^\gamma\}_{\gamma \in \boldsymbol{c}}$ are monomial sets, we obtain the following *truncated moment matrix* of σ:

$$[H_\sigma^{B,C}] \;=\; (\langle \sigma \mid \boldsymbol{x}^{\beta+\gamma} \rangle)_{\gamma \in \boldsymbol{c}, \beta \in \boldsymbol{b}}.$$

Its coefficients depend only on the sum of the indices indexing the rows and columns.

This is a characterization of the classical structure of Hankel matrices when $n = 1$ and $B = \{1, x, x^2, \ldots, x^d\}$, $C = \{1, x, \ldots, x^{d'}\}$ (see e.g. [BP94]). When $n \geqslant 2$, we have a similar family of structured matrices, which rows and columns are indexed by exponents in $\mathbb{N}^n$ (or monolials) and which entries depends on the sum of the row and column indices. These structured matrices called quasi-Hankel matrices have been studied for instance in [MP00].

4.2 Artinian Gorenstein Algebra

In this section, we analyse the properties of Artinian algebras associated to Hankel operators, in the case $m = 1$.

Given $\sigma \in \mathbb{K}[[\boldsymbol{y}]]$, we consider its Hankel operator $H_\sigma : p \in \mathbb{K}[\boldsymbol{x}] \mapsto p \star \sigma \in \mathbb{K}[[\boldsymbol{y}]]$. The kernel I_σ of H_σ is an ideal and the elements $p \star \sigma$ of $\operatorname{im} H_\sigma$ for $p \in \mathbb{K}[\boldsymbol{x}]$ are in $I_\sigma^\perp = \mathscr{A}_\sigma^*$ where $\mathscr{A}_\sigma = \mathbb{K}[\boldsymbol{x}]/I_\sigma$: $\forall q \in I_\sigma$, $\langle p \star \sigma \mid q \rangle = \langle q \star \sigma \mid p \rangle = 0$. If $\mathscr{A}_\sigma$ is artinian of dimension r, then

$$\operatorname{im} H_\sigma = \{p \star \sigma \mid p \in R\} \subset I_\sigma^\perp = \mathscr{A}_\sigma^*$$

is of dimension $\leqslant r$. Therefore, the injective map

$$\begin{array}{rcl} \mathscr{H}_\sigma : \mathscr{A}_\sigma & \to & \mathscr{A}_\sigma^* \\ p(\boldsymbol{x}) & \mapsto & p(\boldsymbol{x}) \star \sigma(\boldsymbol{y}) \end{array}$$

induced by H_σ is an isomorphism, and we have the exact sequence:

$$0 \to I_\sigma \to \mathbb{K}[\boldsymbol{x}] \xrightarrow{H_\sigma} \mathscr{A}_\sigma^* \to 0. \tag{4.1}$$

Proposition 4.2.1 *The inner product $\langle .,. \rangle_\sigma$ is non-degenerate on $\mathscr{A}_\sigma = \mathbb{K}[\boldsymbol{x}]/I_\sigma$.*

Proof. By definition of I_σ, if $p \in \mathbb{K}[\boldsymbol{x}]$ is such that $\forall q \in \mathbb{K}[\boldsymbol{x}]$,

$$\langle p(\boldsymbol{x}), q(\boldsymbol{x}) \rangle_\sigma = \langle p \star \sigma(\boldsymbol{y}) \mid q(\boldsymbol{x}) \rangle = 0,$$

then $p \star \sigma(\boldsymbol{y}) = 0$ and $p \in I_\sigma$. We deduce that the inner product $\langle \cdot, \cdot \rangle_\sigma$ is non-generate on $\mathscr{A}_\sigma = \mathbb{K}[\boldsymbol{x}]/I_\sigma$. □

This is in fact a characterization of Gorenstein Artinian algebra, as shown in the following theorem:

Theorem 4.2.2 *Let $I \subset \mathbb{K}[\boldsymbol{x}]$ be an ideal such that $\mathscr{A} = \mathbb{K}[\boldsymbol{x}]/I$ is Artinian. The following properties are equivalent:*

1. *$\mathscr{A}^\star$ is a free $\mathscr{A}$-module of rank 1 (spanned by $\sigma \in \mathscr{A}^\star$).*
2. *There exists $\sigma \in \mathscr{A}^\star$ such that the inner product $\langle .,. \rangle_\sigma$ is non-degenerate on $\mathscr{A}$.*
3. *There exists an $\mathscr{A}$-isomorphism Δ between $\mathscr{A}^\star$ and $\mathscr{A}$.*
4. *$\mathrm{Hom}_{\mathscr{A}}(\mathscr{A}^\star, \mathscr{A})$ is a free $\mathscr{A}$-module of basis Δ.*

If these properties are satisfied, $\mathscr{A}$ is called a Gorenstein *Artinian algebra.*

(see e.g. [EM07a][chap. 8]).

Example 4.2.3 *Let $I = (x_1^2, x_2^2) \subset \mathbb{K}[x_1, x_2]$. Then $\mathscr{A} = \mathbb{K}[x_1, x_2]/I$ is an Artinian algebra of dimension 4. Its dual is*

$$\mathscr{A}^\star = I^\perp = \langle 1, y_1, y_2, y_1 y_2 \rangle = \mathscr{D}(y_1 y_2).$$

The inverse system $I^\perp = \mathscr{A}^\star$ is generated by the element $y_1 y_2$. It is a free $\mathscr{A}$-module of rank 1, since $p \star y_1 y_2 = 0$ with $p \in \mathbb{K}[x_1, x_2]$ implies that

$$p(0,0) = 0, \partial_1(p)(0,0) = 0, \partial_2(p)(0,0) = 0, \partial_1\partial_2(p)(0,0) = 0.$$

and that $p \in (x_1^2, x_2^2)$ or $p \equiv 0$ in $\mathscr{A}$.

A basis of $\mathscr{A}$ is $B = \{1, x_1, x_2, x_1 x_2\}$. The matrix of $\langle .,. \rangle_{y_1 y_2}$ in this basis is

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

It is invertible and $\langle .,. \rangle_{y_1 y_2}$ is non-degenerate.

These two equivalent properties mean that $\mathscr{A}$ is a Gorenstein Artinian algebra.

Example 4.2.4 *Here is an example of a non-Gorenstein Artinian algebra. Let* $I = (x_1, x_2)^2 \subset \mathbb{K}[x_1, x_2]$*. Then* $\mathscr{A} = \mathbb{K}[x_1, x_2]/I$ *is an Artinian algebra of dimension* 3*. A basis of* $\mathscr{A}$ *is* $B = \{1, x_1, x_2\}$*. Its dual is*

$$\mathscr{A}^\star = I^\perp = \langle 1, y_1, y_2 \rangle = \mathscr{D}(y_1, y_2).$$

The inverse system $I^\perp = \mathscr{A}^\star$ *is generated by the two elements* y_1, y_2*. For any element* $\sigma \in \mathscr{A}^\star$ *of the form* $\sigma = \sigma_0 + \sigma_1 y_1 + \sigma_2 y_2 \in \mathscr{A}^\star$*, with* $\sigma_i \in \mathbb{K}$*, the matrix of* $\langle ., . \rangle_\sigma$ *in the basis* $B = \{1, x_1, x_2\}$ *of* $\mathscr{A}$ *is*

$$\begin{pmatrix} \sigma_0 & \sigma_1 & \sigma_2 \\ \sigma_1 & 0 & 0 \\ \sigma_2 & 0 & 0 \end{pmatrix}.$$

It is of rank $\leqslant 2$*. Thus* $\langle ., . \rangle_\sigma$ *is degenerate and* $\mathscr{A}$ *is not a Gorenstein Artinian algebra.*

Proposition 4.2.1 and Theorem 4.2.2 implies that for $\sigma = \mathbb{K}[\boldsymbol{x}]^\star$ $\sigma \neq 0$, $\sigma \neq 0$, H_σ of finite rank, $I_\sigma = \ker H_\sigma$, the Artinian algebra $\mathscr{A}_\sigma = \mathbb{K}[\boldsymbol{x}]/I_\sigma$ is Gorenstein. Conversely, any Artinian Gorenstein algebra is of this type.

Proposition 4.2.5 *For any Artinian Gorenstein algebra* $\mathscr{A} = \mathbb{K}[\boldsymbol{x}]/I$ *with* I *an ideal of* $\mathbb{K}[\boldsymbol{x}]$*, there exists* $\sigma \in \mathbb{K}[\boldsymbol{x}]^\star$*, such that* $I = \ker H_\sigma$*.*

Proof. As $\mathscr{A} = \mathbb{K}[\boldsymbol{x}]/I$ is Artinian Gorenstein, by Theorem 4.2.2 there exists $\sigma \in \mathscr{A}^\star$ such that σ is a basis of the free $\mathscr{A}$-module $\mathscr{A}^\star$:

$$\mathscr{A}^\star = I^\perp = \sigma \star \mathscr{A}$$

This implies that the map

$$\begin{array}{rcl} H_\sigma : \mathbb{K}[\boldsymbol{x}] & \to & \mathscr{A}^\star \\ p & \mapsto & p \star \sigma \end{array}$$

is surjective and that it induces an isomorphism between $\mathbb{K}[\boldsymbol{x}]/I_\sigma$ and $\mathscr{A}^\star$ where $I_\sigma = \ker H_\sigma$. As $I \subset I_\sigma$ and $\dim \mathscr{A} = \dim \mathscr{A}^\star < \infty$, we deduce that $I = I_\sigma$. □

This construction defines a correspondence between series $\sigma \in \mathbb{K}[[\boldsymbol{y}]]$ of finite rank $\neq 0$ or Hankel operators H_σ of finite rank $\neq 0$ and Artinian Gorenstein Algebras.

4.3 Hankel operators of finite rank

Hankel operators of finite rank play an important role in functional analysis. In one variable and $m = 1$, they are characterized by Kronecker's theorem [Kro80] as follows (see e.g. [Pel98] for more details). Let $\ell_0(\mathbb{K}^\mathbb{N})$ be the vector space of sequences $\in \mathbb{K}^\mathbb{N}$ of finite support and let $\sigma = (\sigma_k)_{k \in \mathbb{N}} \in \mathbb{K}^\mathbb{N}$. The Hankel operator $H_\sigma : (p_l)_{l \in \mathbb{N}} \in \ell_0(\mathbb{K}^\mathbb{N}) \mapsto$

$\left(\sum_l \sigma_{k+l} p_l\right)_{k\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$ is of finite rank r, if and only if, there exist polynomials $\omega_1(u),\ldots,$ $\omega_r(u) \in \mathbb{K}[u]$ and $\xi_1,\ldots,\xi_r \in \mathbb{K}$ distinct such that

$$\sigma_k = \sum_{i=1}^{r} \omega_i(k)\xi_i^k,$$

with $\sum_{i=1}^r \deg(\omega_i) + 1 = \operatorname{rank} H_\sigma$. Rewriting it in terms of generating series, we have $H_\sigma : p = \sum_l p_l x^l \in \mathbb{K}[x] \mapsto \sum_{k\in\mathbb{N}} \left(\sum_l \sigma_{k+l} p_l\right) \frac{y^k}{k!} = p \star \sigma$ is of finite rank, if and only if,

$$\sigma(y) = \sum_{k\in\mathbb{N}} \sigma_k \frac{y^k}{k!} = \sum_{i=1}^{r} \omega_i(y) e^{\xi_i y}$$

with $\omega_1,\ldots,\omega_r \in \mathbb{K}[y]$ and $\xi_1,\ldots,\xi_r \in \mathbb{K}$ distinct such that $\sum_{i=1}^r \deg(\omega_i)+1 = \operatorname{rank} H_\sigma$. Notice that $\deg(\omega_i)+1$ is the dimension of the vector space spanned by $\omega_i(y)$ and all its derivatives.

In the case of several variables, extensions of Kronecker's theorem have been developed [Fli70], [Pow82], [AC16], [AC15], but without connecting the rank of the Hankel operator with the decomposition of the associated symbol. The following result generalizes Kronecker's theorem, by establishing a correspondence between Hankel operators of finite rank and polynomial-exponential series and by connecting the rank of the Hankel operator with the decomposition of the associated series.

Theorem 4.3.1 *Let $\sigma = (\sigma_1,\ldots,\sigma_m) \in (R^\star)^m$. Then $\operatorname{rank} H_\sigma < \infty$, if and only if, $\sigma_k(\mathbf{y}) \in \mathscr{P}ol\mathscr{E}xp(\mathbf{y})$ for $k = 1,\ldots,m$.*

If $\sigma_k(\mathbf{y}) = \sum_{i=1}^r \omega_{k,i}(\mathbf{y})\mathfrak{e}_{\xi_i}(\mathbf{y})$ with $\omega_{k,i}(\mathbf{y}) \in \mathbb{K}[\mathbf{y}]$ and $\xi_i \in \mathbb{K}^n$ pairwise distinct, then $\operatorname{rank} H_\sigma = \sum_{i=1}^r \mu(\omega_{1,i},\ldots,\omega_{m,i})$ where $\mu(\omega_{1,i},\ldots,\omega_{m,i})$ is the dimension of the inverse system $\mathscr{D}(\omega_{1,i},\ldots,\omega_{m,i})$ spanned by $\omega_{k,i}(\mathbf{y})$ and all their derivatives $\partial_{y_1}^{\alpha_1}\cdots\partial_{y_n}^{\alpha_n}\omega_{k,i}(\mathbf{y})$ for $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}^n$, $k = 1,\ldots,m$.

Proof. If H_σ is of finite rank ρ, then $\mathscr{A}_\sigma = \mathbb{K}[\mathbf{x}]/I_\sigma = \mathbb{K}[\mathbf{x}]/\ker H_\sigma \sim \operatorname{Im}(H_\sigma)$ is of dimension ρ and $\mathscr{A}_\sigma$ is an artinian algebra. By Theorem 3.2.1, it can be decomposed as a direct sum of sub-algebras

$$\mathscr{A}_\sigma = \mathscr{A}_{\xi_1} \oplus \cdots \oplus \mathscr{A}_{\xi_r}$$

where $I_\sigma = Q_1 \cap \cdots \cap Q_r$ is a minimal primary decomposition, $\mathscr{V}(I_\sigma) = \{\xi_1,\ldots,\xi_r\}$ and $\mathscr{A}_{\xi_i}$ is the local algebra for the maximal ideal $\mathbf{m}_{\zeta_i}$ defining the root $\xi_i \in \mathbb{K}^n$, such that $\mathscr{A}_{\xi_i} \equiv \mathbb{K}[\mathbf{x}]/Q_i$ where Q_i is a $\mathbf{m}_{\xi_i}$-primary component of I_σ.

By Theorem 3.4.3, for $k = 1,\ldots,m$, the series $\sigma_k \in \mathscr{A}_\sigma^* = I_\sigma^\perp$ can be decomposed as

$$\sigma_k = \sum_{i=1}^{r} \omega_{k,i}(\mathbf{y})\, \mathfrak{e}_{\xi_i}(\mathbf{y}) \tag{4.2}$$

with $\omega_{k,i}(\mathbf{y}) \in \mathbb{K}[\mathbf{y}]$ and $\omega_{k,i}(\mathbf{y})\, \mathfrak{e}_{\xi_i}(\mathbf{y}) \in \mathscr{A}_{\xi_i}^* = Q_i^\perp$, i.e. $\sigma_k \in \mathscr{P}ol\mathscr{E}xp(\mathbf{y})$.

Conversely, let us show that if, for $k = 1,\dots,m$, $\sigma_k(\boldsymbol{y}) = \sum_{i=1}^{r} \omega_{k,i}(\boldsymbol{y})\,\mathfrak{e}_{\xi_i}(\boldsymbol{y})$ with $\omega_i(\boldsymbol{y}) \in \mathbb{K}[\boldsymbol{y}] \setminus \{0\}$ and $\xi_i \in \mathbb{K}^n$ pairwise distinct, the rank of H_σ is finite. Using Lemma 2.2.5, we check that $I_\sigma = \ker H_\sigma$ contains $\cap_{i=1}^{r} \boldsymbol{m}_{\xi_i}^{d_i+1}$ where $d_i = \max_k\{\deg(\omega_{k,i})\}$. Thus $\mathcal{V}(I_\sigma) \subset \{\xi_1,\dots,\xi_r\}$, $\mathcal{A}_\sigma$ is an artinian algebra and $\operatorname{rank} H_\sigma = \dim(\operatorname{Im}(H_\sigma)) = \dim(\mathbb{K}[\boldsymbol{x}]/I_\sigma) = \dim(\mathcal{A}_\sigma) < \infty$.

Let us show now that $\operatorname{rank} H_\sigma = \sum_{i=1}^{r} \mu(\omega_{1,i},\dots,\omega_{m,i})$. By construction,

$$\operatorname{rank} H_\sigma = \dim(R/\ker H_\sigma) = \dim((\ker H_\sigma)^\perp) = \dim((\cap_{k=1}^{m} \ker H_{\sigma_k})^\perp) = \dim(\sum_{k=1}^{m} I_{\sigma_k}^\perp)$$

where $H_{\sigma_k} : p \in R \mapsto p \star \sigma_k \in R^\star$ and $I_{\sigma_k} = \ker H_{\sigma_k}$ for $k = 1,\dots,m$. Consider the Artinian algebra $\mathcal{A}_{\sigma_k} = R/I_{\sigma_k}$ and its decomposition (3.3) as a direct sum of local algebras: $\mathcal{A}_{\sigma_k} = \oplus_{i=1}^{r} \mathcal{A}_{k,\xi_i}$. Thus, we have the dual decomposition:

$$I_{\sigma_k}^\perp = \mathcal{A}_{\sigma_k}^\star = \oplus_{i=1}^{r} \mathcal{A}_{k,\xi_i}^\star.$$

By the exact sequence (4.1), $\mathcal{A}_{\sigma_k}^* = \operatorname{Im}(H_{\sigma_k}) = \{p \star \sigma_k \mid p \in \mathbb{K}[\boldsymbol{x}]\}$. From Lemma 3.4.2, we deduce that $\mathcal{A}_{k,\xi_i}^*$ is spanned by the elements $\boldsymbol{u}_{k,\xi_i} \star (p \star \sigma_k) = p \star (\boldsymbol{u}_{k,\xi_i} \star \sigma_k) = p \star \big(\omega_{k,i}(\boldsymbol{y})\,\mathfrak{e}_{\xi_i}(\boldsymbol{y})\big)$ for $p \in \mathbb{K}[\boldsymbol{x}]$, that is, by $\omega_i(\boldsymbol{y})\,\mathfrak{e}_{\xi_i}(\boldsymbol{y})$ and all its derivatives with respect to $\frac{d}{dy_i}$. This shows that $\mathcal{A}_{\xi_i}^* = \mathcal{D}(\omega_{k,i})\,\mathfrak{e}_{\xi_i}(\boldsymbol{y})$ where $\mathcal{D}(\omega_{k,i}) \subset \mathbb{K}[\boldsymbol{y}]$ is the inverse system spanned by $\omega_{k,i}(\boldsymbol{y})$. It implies that

$$I_\sigma^\perp = \sum_{k=1}^{k} I_{\sigma_k}^\perp = \bigoplus_{i=1}^{r} \left(\sum_{k=1}^{m} \mathcal{D}(\omega_{k,i})\,\mathfrak{e}_{\xi_i} \right) = \bigoplus_{i=1}^{r} \mathcal{D}(\omega_{1,i},\dots,\omega_{m,i})\,\mathfrak{e}_{\xi_i}$$

We deduce that $\operatorname{rank} H_\sigma = \sum_{i=1}^{r} \dim\big(\mathcal{D}(\omega_{1,i},\dots,\omega_{m,i})\big) = \sum_{i=1}^{r} \mu(\omega_{1,i},\dots,\omega_{m,i})$. This concludes the proof of the theorem. □

Here are some direct consequences of this result.

Proposition 4.3.2 *If $\sigma = (\sigma_1,\dots,\sigma_m)$ with $\sigma_k(\boldsymbol{y}) = \sum_{i=1}^{r} \omega_{k,i}(\boldsymbol{y})\mathfrak{e}_{\xi_i}(\boldsymbol{y})$ and $\omega_{k,i}(\boldsymbol{y}) \in \mathbb{K}[\boldsymbol{y}]$ not all zero for $k = 1,\dots,m$ and $\xi_i \in \mathbb{K}^n$ pairwise distinct, then we have the following properties:*

- *The points $\xi_1,\xi_2,\dots,\xi_r \in \mathbb{K}^n$ are the common roots of the polynomials in $I_\sigma = \ker H_\sigma = \{p \in \mathbb{K}[\boldsymbol{x}] \mid \forall q \in \mathbb{K}[\boldsymbol{x}], \langle \sigma | pq \rangle = 0\}$.*
- *The inverse system of Q_i is $\mathcal{D}(\omega_{1,i},\dots,\omega_{m,i})$, where Q_i is the primary component of I_σ associated to ξ_i.*

Proof. From the previous proof of Theorem 4.3.1, we have

$$I_\sigma^\perp = \bigoplus_{i=1}^{r} Q_i^\perp = \bigoplus_{i=1}^{r} \mathcal{D}(\omega_{1,i},\dots,\omega_{m,i})\,\mathfrak{e}_{\xi_i}$$

with $Q_i^\perp = \mathcal{D}(\omega_{1,i},\dots,\omega_{m,i})\,\mathfrak{e}_{\xi_i}$. This shows that

- Q_i is $\boldsymbol{m}_{\xi_i}$-primary and $\mathcal{V}(I_\sigma) = \{\xi_1, \ldots, \xi_r\}$,
- the inverse system of Q_i is $\mathscr{D}(\omega_{1,i}, \ldots, \omega_{m,i})\, \mathfrak{e}_{\xi_i}$.

□

A special case of interest is when the roots are simple. We characterize it as follows:

Proposition 4.3.3 *Let $\sigma(\boldsymbol{y}) \in (\mathbb{K}[[\boldsymbol{y}]])^m$. The following conditions are equivalent:*

1. *$\sigma_k(\boldsymbol{y}) = \sum_{i=1}^{r} \omega_{k,i}\, \mathfrak{e}_{\xi_i}(\boldsymbol{y})$, with $\{\omega_{1,i}, \ldots, \omega_{m,i}\} \subset \mathbb{K}$ not all zero and $\xi_i \in \mathbb{K}^n$ pairwise distinct.*
2. *The rank of H_σ is $r = \#\mathcal{V}(I_\sigma)$ and the multiplicity of the roots $\xi_1, \ldots, \xi_r \in \mathcal{V}(I_\sigma)$ is 1.*
3. *A basis of $\mathscr{A}_\sigma^*$ is $\mathfrak{e}_{\xi_1}, \ldots, \mathfrak{e}_{\xi_r}$.*

Proof. $1 \Rightarrow 2$. By theorem 4.3.1, the rank of H_σ is $\sum_{i=1}^{r} \mu(\omega_{1,i}, \ldots, \omega_{m,i})$ the dimension of the space spanned by $\omega_{k,i}$ and their derivatives. As $\{\omega_{1,i}, \ldots, \omega_{m,i}\} \subset \mathbb{K}$ are not all zero, $\operatorname{rank} H_\sigma = r$ and the multiplicity of the root ξ_i is $\mu(\omega_{1,i}, \ldots, \omega_{m,i}) = 1$.

$2 \Rightarrow 3$. As the multiplicity of the roots ξ_i is 1 and $\sigma_k \in I_\sigma^\perp$, by Theorem 3.4.3 $\sigma_k = \sum_{i=1}^{r} \omega_{k,i}\, \mathfrak{e}_{\xi_i}$ with $\omega_{k,i} = 0$ or $\deg(\omega_{k,i}) = 0$. By Theorem 4.3.1, we have

$$\mathscr{A}_\sigma^* = I_\sigma^\perp = \bigoplus_{i=1}^{r} \mathscr{D}(\omega_{1,i}, \ldots, \omega_{m,i})\mathfrak{e}_{\xi_i} = \bigoplus_{i=1}^{r} \mathbb{K}\mathfrak{e}_{\xi_i}.$$

This shows that $\mathfrak{e}_{\xi_1}, \ldots, \mathfrak{e}_{\xi_r}$ is a basis of $\mathscr{A}_\sigma^*$.

$3 \Rightarrow 1$. As $\mathfrak{e}_{\xi_1}, \ldots, \mathfrak{e}_{\xi_r}$ is a basis $\mathscr{A}_\sigma^\star$, the points $\xi_i \in \mathbb{K}^n$ are pairwise distinct. As $\sigma_k \in \mathscr{A}_\sigma^\star$, there exists $\omega_{k,i} \in \mathbb{K}$ such that $\sigma_k = \sum_{i=1}^{r} \omega_{k,i}\mathfrak{e}_{\xi_i}$. If all the coefficients $\omega_{1,i}, \ldots, \omega_{m,i}$ vanish then $\dim(\mathscr{A}_\sigma^\star) < r$, which is contradicting point 3. Thus $\omega_{1,i}, \ldots, \omega_{m,i}$ are not all zero. □

Given a Hankel operator H_σ of finite rank r, it is clear that the truncated operators will have at most rank r. We have a converse property, so-called *flat extension* property, which gives conditions under which a truncated Hankel operator of rank r can be extended to a Hankel operator of the same rank (see [LM09] and extensions [BCMT10], [BBCM13], [Mou16]).

Theorem 4.3.4 *Let $V, V' \subset \mathbb{K}[\boldsymbol{x}]$ be vector spaces connected to 1, such that $x_1, \ldots, x_n \in V$ and let $\sigma \in \langle V \cdot V' \rangle^*$. Let $B \subset V$, $B' \subset V'$ such that $B^+ \subset V, B'^+ \subset V'$. If $\operatorname{rank} H_\sigma^{V,V'} = \operatorname{rank} H_\sigma^{B,B'} = r$, then there is a unique extension $\tilde{\sigma} \in \mathbb{K}[[\boldsymbol{y}]]$ such that $\tilde{\sigma}$ coincides with σ on $\langle V \cdot V' \rangle$ and $\operatorname{rank} H_{\tilde{\sigma}} = r$. In this case, $\tilde{\sigma} \in \mathscr{POLYEXP}$ with $r = \mu(\tilde{\sigma})$ and $I_{\tilde{\sigma}} = (\ker H_\sigma^{B^+,B'})$.*

We will use this property in a decomposition method to test when to stop (see section 4.6).

4.4 Decomposition of series

The sparse decomposition problem of series $\sigma \in \mathbb{K}[[\boldsymbol{y}]]$ consists in computing points $\{\xi_1, \ldots, \xi_r\} \subset \mathbb{K}^n$ and weights $\omega_i(\boldsymbol{y}) \in \mathbb{K}[\boldsymbol{y}]$ such that $\sigma = \sum_{i=1}^{r} \omega_i(\boldsymbol{y})\, \mathfrak{e}_{\xi_i}(\boldsymbol{y})$. In this section, we describe how to compute this decomposition from the Hankel operator H_σ.

We recall classical results on the resolution of polynomial equations by eigenvalue and eigenvector computation, that we will use to compute the decomposition. Hereafter, $\mathscr{A} = \mathbb{K}[\boldsymbol{x}]/I$ is the quotient algebra of $\mathbb{K}[\boldsymbol{x}]$ by any ideal I and $\mathscr{A}^* = \mathrm{Hom}_{\mathbb{K}}(\mathscr{A}, \mathbb{K})$ is the dual of $\mathscr{A}$. It is naturally identified with the orthogonal $I^\perp = \{\Lambda \in \mathbb{K}[[\boldsymbol{y}]] \mid \forall p \in I, \langle \Lambda, p\rangle = 0\}$. In the reconstruction problem, we will take $I = I_\sigma$.

In the first step of method, we will determine a basis of $\mathscr{A}_\sigma = \mathbb{K}[\boldsymbol{x}]/I_\sigma$. We will use the following result:

Lemma 4.4.1 *Let $B = \{b_1, \ldots, b_r\} \subset \mathbb{K}[\boldsymbol{x}]$, $C = \{C_1, \ldots, C_m\} \subset (\mathbb{K}[\boldsymbol{x}])^m$. If the matrix $H_\sigma^{B,C} = (\langle \sigma_k | b_j c_{i,k}\rangle)_{c_{i,k} \in C, b_j \in B}$ is invertible, then B is linearly independent in $\mathscr{A}_\sigma$.*

Proof. Suppose that $H_\sigma^{B,C}$ is invertible. If there exist $p = \sum_j p_j b_j$ ($p_j \in \mathbb{K}$) such that $p \equiv 0$ in $\mathscr{A}_\sigma$. Then $p \star \sigma = 0$ and $\forall q \in R$, $k = 1, \ldots, m$ $\langle \sigma_k | pq\rangle = 0$. In particular, we have

$$\sum_{j=1}^{r} \langle \sigma | b_j c_{k,j}\rangle p_j = 0.$$

As $H_\sigma^{B,C}$ is invertible, $p_j = 0$ for $j = 1, \ldots, r$ and B is a family of linearly independent elements in $\mathscr{A}_\sigma$. □

Notice that this result depend only on the classes of b_j, $c_{k,j}$ modulo I_σ (i.e. in $\mathscr{A}_\sigma$) since

$$\langle \sigma_k | (b_i + p)(c_{k,j} + p')\rangle = \langle \sigma_k | b_i c_{k,j}\rangle$$

for any $p, p' \in I_\sigma$.

The converse of Lemma 4.4.1 is not necessarily true, as shown by the following example in one variable: if $m = 1$, $\sigma = y$, then $I_\sigma = (x^2)$, $\mathscr{A}_\sigma = \mathbb{K}[x]/(x^2)$ and $B = C = \{1\}$ are linearly independent in $\mathscr{A}_\sigma$, but $H_\sigma^{B,C} = (\langle \sigma | 1\rangle) = (0)$ is not invertible.

This lemma implies that if $\dim \mathscr{A}_\sigma < +\infty$, $|B| = |C| = \dim \mathscr{A}_\sigma$ and $H_\sigma^{B,C}$ is invertible, then B is a basis of $\mathscr{A}_\sigma$.

If $m = 1$ and $B, C \subset \mathbb{K}[\boldsymbol{x}]$ such that $|B| = |C|$ and $H_\sigma^{B,C}$ invertible, we can also deduce from Lemma 4.4.1 that C is linearly independent in $\mathscr{A}_\sigma$ since $H_\sigma^{C,B} = (H_\sigma^{B,C})^t$ is invertible.

By quotient by $I_\sigma = \ker H_\sigma$, the Hankel operator H_σ induces the map

$$\begin{aligned} \mathscr{H}_\sigma : \mathscr{A}_\sigma &\to (\mathscr{A}_\sigma^\star)^m \\ p &\mapsto p \star \sigma. \end{aligned}$$

For $g \in \mathbb{K}[\boldsymbol{x}]$, the operator of multiplication by g in $\mathscr{A}_\sigma$ is

$$\begin{aligned} \mathscr{M}_g : \mathscr{A}_\sigma &\to \mathscr{A}_\sigma \\ p &\mapsto g\, p. \end{aligned}$$

Lemma 4.4.2 *For any $g \in \mathbb{K}[\boldsymbol{x}]$, we have*

$$\mathscr{H}_{g\star\sigma} = \mathscr{H}_\sigma \circ \mathscr{M}_g. \tag{4.3}$$

Proof. This is a direct consequence of the definitions of $\mathscr{H}_{g\star\sigma}, \mathscr{H}_\sigma$ and $\mathscr{M}_g$. □

The transpose operator of multiplication by g is (by definition of the tranposition)

$$\begin{array}{rcl} \mathscr{M}_g^t : \mathscr{A}_\sigma^\star & \to & \mathscr{A}_\sigma^\star \\ \sigma & \mapsto & g\star\sigma. \end{array}$$

When $m=1$, we have an additional relation:

Lemma 4.4.3 *For any $g \in \mathbb{K}[\boldsymbol{x}]$ and $\sigma \in \mathbb{K}[\boldsymbol{x}]^\star$, we have*

$$\mathscr{H}_{g\star\sigma} = \mathscr{M}_g^t \circ \mathscr{H}_\sigma. \tag{4.4}$$

Proof. This is also a direct consequence of the commutativity of the product in $\mathscr{A}_\sigma$ and the definitions of $\mathscr{H}_{g\star\sigma}, \mathscr{H}_\sigma$ and $\mathscr{M}_g^t$. □

From Relation (4.3) and Proposition 3.3.2, we have the following property.

Proposition 4.4.4 *If $\sigma(\boldsymbol{y}) = \sum_{i=1}^r \omega_i(\boldsymbol{y})\mathfrak{e}_{\xi_i}(\boldsymbol{y})$with $\omega_i \in \mathbb{K}[\boldsymbol{y}]\setminus\{0\}$ and $\xi_i \in \mathbb{K}^n$ distinct, then*

- *for all $g \in \mathscr{A}$, the generalized eigenvalues of $(\mathscr{H}_{g\star\sigma}, \mathscr{H}_\sigma)$ are $g(\xi_i)$ with multiplicity $\mu_i = \mu(\omega_i)$, $i = 1\ldots r$,*
- *the generalized eigenvectors common to all $(\mathscr{H}_{g\star\sigma}, \mathscr{H}_\sigma)$ with $g \in \mathscr{A}$ are - up to a scalar - $\mathscr{H}_\sigma^{-1}(\mathfrak{e}_{\xi_1}), \ldots, \mathscr{H}_\sigma^{-1}(\mathfrak{e}_{\xi_r})$.*

Remark 4.4.5 *If we take $g = x_i$, then the eigenvalues are the i-th coordinates of the points ξ_j.*

4.4.1 The case of simple roots

We consider the case where $m = 1$ and I_σ defines simple roots, that is σ is of the form $\sigma(\boldsymbol{y}) = \sum_{i=1}^r \omega_i \mathfrak{e}_{\xi_i}(\boldsymbol{y})$ with $\omega_i \in \mathbb{K}\setminus\{0\}$ and $\xi_i \in \mathbb{K}^n$ distinct, computing the decomposition reduces to a simple eigenvector computation, as we will see.

By Proposition 4.3.3, $\{\mathfrak{e}_{\xi_1}, \ldots, \mathfrak{e}_{\xi_r}\}$ is a basis of $\mathscr{A}_\sigma^*$. We denote by $\{\boldsymbol{u}_{\xi_1}, \ldots, \boldsymbol{u}_{\xi_r}\}$ the basis of $\mathscr{A}_\sigma$, which is dual to $\{\mathfrak{e}_{\xi_1}, \ldots, \mathfrak{e}_{\xi_r}\}$, so that $\forall a \in \mathscr{A}_\sigma$,

$$a(\boldsymbol{x}) \equiv \sum_{i=1}^r \langle \mathfrak{e}_{\xi_i} \mid a\rangle\, \boldsymbol{u}_{\xi_i}(\boldsymbol{x}) \equiv \sum_{i=1}^r a(\xi_i)\, \boldsymbol{u}_{\xi_i}(\boldsymbol{x}). \tag{4.5}$$

From this formula, we easily verify that the polynomials $\boldsymbol{u}_{\xi_1}, \boldsymbol{u}_{\xi_2}, \ldots, \boldsymbol{u}_{\xi_r}$ are the *interpolation polynomials* at the points $\xi_1, \xi_2, \ldots, \xi_r$, and satisfy the following relations in $\mathscr{A}_\sigma$:

- $\boldsymbol{u}_{\xi_i}(\xi_j)=\begin{cases} 1 & \text{if } i=j, \\ 0 & \text{otherwise}. \end{cases}$
- $\boldsymbol{u}_{\xi_i}(\boldsymbol{x})^2 \equiv \boldsymbol{u}_{\xi_i}(\boldsymbol{x})$.
- $\sum_{i=1}^{r} \mathbf{u}_{\xi_i}(\boldsymbol{x}) \equiv 1$.

Proposition 4.4.6 *Let $\sigma = \sum_{i=1}^{r} \omega_i \mathfrak{e}_{\xi_i}(\boldsymbol{y})$ with ξ_i pairwise distinct and $\omega_i \in \mathbb{K} \setminus \{0\}$. The basis $\{\boldsymbol{u}_{\xi_1}, \ldots, \boldsymbol{u}_{\xi_r}\}$ is an orthogonal basis of $\mathscr{A}_\sigma$ for the inner product $\langle .,. \rangle_\sigma$ and satisfies $\langle \boldsymbol{u}_{\xi_i}, 1\rangle_\sigma = \langle \sigma \mid \boldsymbol{u}_{\xi_i}\rangle = \omega_i$ for $i = 1 \ldots, r$.*

Proof. For $i, j = 1 \ldots r$, we have $\langle \boldsymbol{u}_{\xi_i}, \boldsymbol{u}_{\xi_j}\rangle_\sigma = \langle \sigma \mid \boldsymbol{u}_{\xi_i}\boldsymbol{u}_{\xi_j}\rangle = \sum_{k=1}^{r} \omega_k \boldsymbol{u}_{\xi_i}(\xi_k)\boldsymbol{u}_{\xi_j}(\xi_k)$. Thus

$$\langle \boldsymbol{u}_{\xi_i}, \boldsymbol{u}_{\xi_j}\rangle_\sigma = \begin{cases} \omega_i \text{ if } i=j \\ 0 \text{ otherwise} \end{cases}$$

and $\{\boldsymbol{u}_{\xi_1}, \ldots, \boldsymbol{u}_{\xi_r}\}$ is an orthogonal basis of $\mathscr{A}_\sigma$. Moreover,

$$\langle \boldsymbol{u}_{\xi_i}, 1\rangle_\sigma = \langle \sigma \mid \boldsymbol{u}_{\xi_i}\rangle = \sum_{k=1}^{r} \omega_k \boldsymbol{u}_{\xi_i}(\xi_k) = \omega_i.$$

□

Proposition 3.3.2 implies the following result:

Corollary 4.4.7 *If $g \in \mathbb{K}[\boldsymbol{x}]$ is separating the roots $\xi_1, \ldots, \xi_r$ (i.e. $g(\xi_i) \neq g(\xi_j)$ when $i \neq j$), then*

- *the operator $\mathscr{M}_g$ is diagonalizable and its eigenvalues are $g(\xi_1), \ldots, g(\xi_r)$,*
- *the corresponding eigenvectors of $\mathscr{M}_g$ are, up to a non-zero scalar, the interpolation polynomials $\boldsymbol{u}_{\xi_1}, \ldots, \boldsymbol{u}_{\xi_r}$.*
- *the corresponding eigenvectors of $\mathscr{M}_g^t$ are, up to a non-zero scalar, the evaluations $\mathfrak{e}_{\xi_1}, \ldots, \mathfrak{e}_{\xi_r}$.*

A simple computation shows that $H_\sigma(\boldsymbol{u}_{\xi_i}) = \omega_i \mathfrak{e}_{\xi_i}(\boldsymbol{y})$ for $i = 1, \ldots, r$. This leads to the following formula for the weights of the decomposition of σ:

Proposition 4.4.8 *If $\sigma = \sum_{i=1}^{r} \omega_i \mathfrak{e}_{\xi_i}(\boldsymbol{y})$ with ξ_i pairwise distinct and $\omega_i \in \mathbb{K} \setminus \{0\}$ and $g \in \mathbb{K}[\boldsymbol{x}]$ is separating the roots $\xi_1, \ldots, \xi_r$, then there are r linearly independent generalized eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ of $(\mathscr{H}_{g\star\sigma}, \mathscr{H}_\sigma)$, which satisfy the relations:*

$$\begin{aligned} \langle \sigma \mid x_j \boldsymbol{v}_i\rangle &= \xi_{i,j}\langle \sigma \mid \boldsymbol{v}_i\rangle \text{ for } j = 1, \ldots, n, i = 1, \ldots, r \\ \sigma(\boldsymbol{y}) &= \sum_{i=1}^{r} \frac{1}{\boldsymbol{v}_i(\xi_i)} \langle \sigma \mid \boldsymbol{v}_i\rangle\, \mathfrak{e}_{\xi_i}(\boldsymbol{y}) \end{aligned}$$

Proof. By Lemma 4.4.2 and Corollary 4.4.7, the eigenvectors $\boldsymbol{u}_{\xi_1},\ldots,\boldsymbol{u}_{\xi_r}$ of $\mathcal{M}_g$ are the generalized eigenvectors of $(\mathcal{H}_{g\star\sigma},\mathcal{H}_\sigma)$. By Corollary 4.4.7, $\boldsymbol{v}_i$ is a multiple of the interpolation polynomial $\boldsymbol{u}_{\xi_i}$, and thus of the form $\boldsymbol{v}_i(\boldsymbol{x})=\boldsymbol{v}_i(\xi_i)\boldsymbol{u}_{\xi_i}(\boldsymbol{x})$ since $\boldsymbol{u}_{\xi_i}(\xi_i)$=1. We deduce that $\boldsymbol{u}_{\xi_i}(\boldsymbol{x})=\frac{1}{\boldsymbol{v}_i(\xi_i)}\boldsymbol{v}_i(\boldsymbol{x})$. By Proposition 4.4.6, we have

$$\omega_i=\langle\sigma\mid\boldsymbol{u}_{\xi_i}\rangle=\frac{1}{\boldsymbol{v}_i(\xi_i)}\langle\sigma\mid\boldsymbol{v}_i\rangle,$$

from which, we deduce the decomposition of $\sigma=\sum_{i=1}^r\frac{1}{\boldsymbol{v}_i(\xi_i)}\langle\sigma\mid\boldsymbol{v}_i\rangle\,\mathfrak{e}_{\xi_i}(\boldsymbol{y})$. It implies that

$$\langle\sigma\mid x_j\boldsymbol{u}_{\xi_i}\rangle=\sum_{k=1}^r\omega_k\xi_{k,j}\boldsymbol{u}_{\xi_i}(\xi_k)=\xi_{i,j}\omega_i=\xi_{i,j}\langle\sigma\mid\boldsymbol{u}_{\xi_i}\rangle.$$

Multiplying by $\boldsymbol{v}_i(\xi_i)$, we obtain the first relations. □

4.4.2 The case of multiple roots

We consider now the more general case where $m=1$ and σ is of the form

$$\sigma=\sum_{i=1}^r\omega_i(\boldsymbol{y})\mathfrak{e}_{\xi_i}(\boldsymbol{y})$$

with $\omega_i(\boldsymbol{y})\in\mathbb{K}[\boldsymbol{y}]$ and $\xi_i\in\mathbb{K}^n$ pairwise distinct. By Theorem 3.2.1, we have

$$\mathcal{A}_\sigma=\mathcal{A}_{\sigma,\xi_1}\oplus\cdots\oplus\mathcal{A}_{\sigma,\xi_r}$$

where $\mathcal{A}_{\sigma,\xi_i}\simeq\mathbb{K}[\boldsymbol{x}]/Q_i$ is the local algebra associated to the $\mathfrak{m}_{\xi_i}$-primary component Q_i of I_σ. The decomposition (3.3) and Proposition 3.4.1 imply that $\mathcal{A}_{\sigma,\xi_i}$ is a local Artinian Gorenstein Algebra such that $\boldsymbol{u}_{\xi_i}\star\sigma$ is a basis of $\mathcal{A}^*_{\sigma,\xi_i}$. The operators $\mathcal{M}_{x_j}$ of multiplication by the variables x_j in $\mathcal{A}_\sigma$ for $j=1,\ldots,n$ are commuting and have a block diagonal decomposition, corresponding to the decomposition of $\mathcal{A}_\sigma$.

It turns out that the operators $\mathcal{M}_{x_j}$ have common eigenvectors $\boldsymbol{v}_i(\boldsymbol{x})\in\mathcal{A}_{\sigma,\xi_i}$. Such an eigenvector is an element of the *socle* $(0:\mathfrak{m}_{\xi_i})=\{v\in\mathcal{A}_{\sigma,\xi_i}\mid(x_j-\xi_{i,j})\boldsymbol{v}\equiv 0, j=1,\ldots,n\}=(Q_i:\mathfrak{m}_{\xi_i})/Q_i$.

In the case of an Artinian Gorenstein algebra $\mathcal{A}_{\sigma,\xi_i}$, the socle $(0:\mathfrak{m}_{\xi_i})$ is a vector space of dimension 1 (see e.g. [EM07b] [Sec. 7.1.5 and Sec. 9.5] for a simple proof). A basis element can be computed as a common eigenvector of the commuting operators $\mathcal{M}_{x_j}$. The corresponding eigenvalues are the coordinates $\xi_{i,1},\ldots,\xi_{i,n}$ of the roots ξ_i, $i=1,\ldots,r$.

For a separating linear form $\boldsymbol{l}(\boldsymbol{x})=l_1x_1+\cdots+l_nx_n$ (such that $\boldsymbol{l}(\xi_i)\neq\boldsymbol{l}(\xi_j)$ if $i\neq j$), the eigenspace of $\mathcal{M}_{\boldsymbol{l}}$ for the eigenvalue $\boldsymbol{l}(\xi_i)$ is the local algebra $\mathcal{A}_{\xi_i}$ associated to the root ξ_i. Let $B_i=\{b_{i,1},\ldots,b_{i,\mu_i}\}$ be a basis of this eigenspace. It spans the elements of $\mathcal{A}_{\xi_i}$, which are of the form $\boldsymbol{u}_{\xi_i}a$ for $a\in\mathcal{A}_\sigma$ where $\boldsymbol{u}_{\xi_i}$ is the idempotent associated to ξ_i

(see Theorem 3.2.1). In particular, the eigenspace of $\mathcal{M}_{\boldsymbol{l}(\boldsymbol{x})}$ associated to the eigenvalue $\boldsymbol{l}(\xi_i)$ contains the idempotent $\boldsymbol{u}_{\xi_i}$, which can be recovered as follows:

Lemma 4.4.9 *Let $B_i = \{b_{i,1}, \ldots, b_{i,\mu_i}\}$ be a basis of $\mathcal{A}_{\xi_i}$ and $\mathbf{U_i} = (\langle \sigma \mid b_{i,k}\rangle)_{k=1,\ldots,\mu_i}$. Then $(H_\sigma^{B_i,B_i})^{-1}U_i$ is the coefficient vector of the idempotent $\boldsymbol{u}_{\xi_i}$ in the basis B_i of $\mathcal{A}_{\xi_i}$.*

Proof. As the idempotent $\boldsymbol{u}_{\xi_i}$ satisfies the relation $\boldsymbol{u}_{\xi_i}^2 \equiv \boldsymbol{u}_{\xi_i}$ in $\mathcal{A}_\sigma$ and $\mathcal{A}_{\xi_i} = \boldsymbol{u}_{\xi_i}\mathcal{A}_\sigma$, we have

$$\langle \boldsymbol{u}_{\xi_i} \star \sigma \mid b_{i,k}\rangle = \langle \sigma \mid \boldsymbol{u}_{\xi_i} b_{i,k}\rangle = \langle \sigma \mid b_{i,k}\rangle,$$

and $\mathbf{U_i} = (\langle \sigma \mid \rangle b_{i,k})_{k=1,\ldots,\mu_i}$ is the coefficient vector of $\boldsymbol{u}_{\xi_i}\star\sigma$ in the dual basis of B_i in $\mathcal{A}_{\xi_i}^*$. By Lemma 4.4.1, as B_i is a basis of $\mathcal{A}_{\xi_i}$, $H_\sigma^{B_i,B_i}$ is invertible and $(H_\sigma^{B_i,B_i})^{-1}U_i$ is the coefficient vector of $\boldsymbol{u}_{\xi_i}$ in the basis B_i of $\mathcal{A}_{\xi_i}$. □

Using the idempotent $\boldsymbol{u}_{\xi_i}$, we have the following formula for the weights $\omega_i(\boldsymbol{y})$ in the decomposition of σ:

Proposition 4.4.10 *The polynomial coefficient of $\boldsymbol{e}_{\xi_i}(\boldsymbol{y})$ in the decomposition of σ is*

$$\omega_i(\boldsymbol{y}) = \sum_{\alpha\in\mathbb{N}^n} \langle \boldsymbol{u}_{\xi_i} \star \sigma \mid (\boldsymbol{x} - \xi_i)^\alpha\rangle \frac{\boldsymbol{y}^\alpha}{\alpha!}. \tag{4.6}$$

Proof. By Theorem 4.2.2 and relation (3.3), we have

$$\boldsymbol{u}_{\xi_i} \star \sigma = \omega_i(\boldsymbol{y})\boldsymbol{e}_{\xi_i}(\boldsymbol{y}).$$

As $\langle \boldsymbol{y}^\beta \mathfrak{e}_{\xi_i}(\boldsymbol{y}) \mid (\boldsymbol{x} - \xi_i)^\alpha\rangle = \left\{ \begin{array}{ll} \alpha! & \text{if } \alpha = \beta \\ 0 & \text{otherwise} \end{array}\right.$, we deduce the decomposition (4.6), which is a finite sum since $\omega_i(\boldsymbol{y}) \in \mathbb{K}[\boldsymbol{y}]$. □

4.5 Decomposition algorithm

The results of the previous section show that the decomposition of σ can be deduced from the generalized eigenvectors of $(\mathcal{H}_{g\star\sigma}, \mathcal{H}_\sigma)$.

This is summarized in the following algorithm, which computes the decomposition of σ, assuming a basis B of $\mathcal{A}_\sigma$ is known.

Algorithm 4.5.1: Decomposition of polynomial-exponential series

Input: the (first) coefficients σ_α of a series $\sigma \in \mathbb{K}[[\boldsymbol{y}]]$ for $\alpha \in \boldsymbol{a} \subset \mathbb{N}^n$ and bases $B = \{b_1, \ldots, b_r\}$, $B = \{b'_1, \ldots, b'_r\}$, of $\mathscr{A}_\sigma$ such that $\langle B' \cdot B^+ \rangle \subset \langle \boldsymbol{x}^{\boldsymbol{a}} \rangle$.

1. Construct the matrices $H_0 = (\langle \sigma \mid b'_i b_j \rangle)_{1 \leqslant i,j \leqslant r}$ (resp. $H_k = (\langle \sigma \mid x_k b'_i b_j \rangle)_{1 \leqslant i,j \leqslant r}$) of $\mathscr{H}_\sigma$ (resp. $\mathscr{H}_{x_k \star \sigma}$) in the basis B of $\mathscr{A}_\sigma$;
2. Take a separating linear form $\boldsymbol{l}(\boldsymbol{x}) = l_1 x_1 + \cdots + l_n x_n$ and construct $H_{\boldsymbol{l}} = \sum_{i=1}^n l_i H_i = (\langle \sigma \mid \boldsymbol{l}\, b'_i b_j \rangle)_{1 \leqslant i,j \leqslant r}$;
3. Compute bases $B_i, i = 1, \ldots, r'$ of the generalized eigenspaces of $(H_{\boldsymbol{l}}, H_0)$;
4. For each basis $B_i = \{b_{i,1}, \ldots, b_{i,\mu_i}\}$, compute $\mathbf{U}_i = (\langle \sigma \mid b_{i,k} \rangle)_{k=1,\ldots,\mu_i}$ and $\boldsymbol{u}_i = (H_\sigma^{B_i,B_i})^{-1} \mathbf{U}_i$;
5. Compute common eigenvectors $\boldsymbol{v}_i \in \langle B_i \rangle$ $i = 1, \ldots, r'$ of all the pencils (H_k, H_0), $k = 1, \ldots, n$ and $\xi_i = (\xi_{i,1}, \ldots, \xi_{i,n})$ such that $(H_k - \xi_{i,k} H_0)\boldsymbol{v}_i = 0$;
6. Compute $\omega_i(\boldsymbol{y}) = \sum_{\alpha \in \mathbb{N}^n} \langle \boldsymbol{u}_i \star \sigma \mid (\boldsymbol{x} - \xi_i)^\alpha \rangle \frac{\boldsymbol{y}^\alpha}{\alpha!}$;

Output: the decomposition $\sigma(\boldsymbol{y}) = \sum_{i=1}^r \omega_i(\boldsymbol{y}) \mathfrak{e}_{\xi_i}(\boldsymbol{y})$.

To apply this algorithm, one need to compute a basis B of $\mathscr{A}_\sigma$ such that σ is defined on $B \cdot B^+$ where $B^+ = \cup_{i=1}^n x_i B \cup B$. In Section 4.6, we will detail an efficient method to compute such a basis B and a characterization of the sequences $(\sigma_\alpha)_{\alpha \in A}$, which admits a decomposition of rank r.

The second step of the algorithm consists in taking a linear form $\boldsymbol{l}(\boldsymbol{x}) = l_1 x_1 + \cdots + l_n x_n$, which separates the roots in the decomposition ($\boldsymbol{l}(\xi_i) \neq \boldsymbol{l}(\xi_j)$ if $i \neq j$). A generic choice of $\boldsymbol{l}$ yields a separating linear form. This separating property can be verified a posteriori, by checking that there are r distinct generalized eigenvalues. Notice that we only need to compute the matrix $H_{\boldsymbol{l}}$ of $\mathscr{H}_{\boldsymbol{l} \star \sigma}$ in the basis B of $\mathscr{A}_\sigma$ and not necessarily all the matrices H_k.

The third step is the computation of generalized eigenvectors of a Hankel pencil. The other steps involve the application of σ on polynomials in B^+.

The fifth step computes eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ common to all the pencil of matrices. Efficient methods as in [GT09] can be used to computed them from $\langle B_i \rangle$ when the eigenvalue is not simple. In the case of a simple eigenvalue, step 5 can be removed since the vector $\boldsymbol{u}_i$ computed in step 4 or the element $b_{i,1}$ in the basis B'_i is a common eigenvector.

In step 6, only a finite number of terms $\langle \boldsymbol{u}_i \star \sigma | (\boldsymbol{x} - \xi_i)^\alpha \rangle$ need to be computed. If the weight ω_i is are constant, its computation in the last step can be replaced by $\omega_i = \langle \sigma | \boldsymbol{u}_i \rangle$.

Notice that the weights ω_i are recovered directly from the polynomials $\boldsymbol{u}_i$ and that it is not necessary to solve a Vandermonde linear system to compute them as in the pencil method (see Section 1.1).

4.5.1 Example

We illustrate the method on a sequence σ_α obtained by evaluation of a sum of exponentials on a grid.

We consider the function $h(u_1, u_2) = 2 + 3 \cdot 2^{u_1} 2^{u_2} - 3^{u_1}$. Its associated generating series is $\sigma = \sum_{\alpha \in \mathbb{N}^2} h(\alpha) \frac{y^\alpha}{\alpha!}$. Its (truncated) moment matrix is

$$H_\sigma^{[1,x_1,x_2,x_1^2,x_1x_2,x_2^2]} = \begin{bmatrix} h(0,0) & h(1,0) & h(0,1) & \cdots \\ h(1,0) & h(2,0) & h(1,1) & \cdots \\ h(0,1) & h(1,1) & h(0,2) & \cdots \\ \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \end{bmatrix} = \begin{bmatrix} 4 & 5 & 7 & 5 & 11 & 13 \\ 5 & 5 & 11 & -1 & 17 & 23 \\ 7 & 11 & 13 & 17 & 23 & 25 \\ 5 & -1 & 17 & -31 & 23 & 41 \\ 11 & 17 & 23 & 23 & 41 & 47 \\ 13 & 23 & 25 & 41 & 47 & 49 \end{bmatrix}$$

We compute $B = \{1, x_1, x_2\}$. The generalized eigenvalues of $(H_{x_1 \star \sigma}, H_\sigma)$ are $[1, 2, 3]$ and corresponding eigenvectors are represented by the columns of

$$\boldsymbol{u} := \begin{bmatrix} 2 & -1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix},$$

associated to the polynomials $\boldsymbol{u}(x) = [2 - \frac{1}{2}\, x_1 - \frac{1}{2}\, x_2, -1 + x_2, \frac{1}{2}\, x_1 - \frac{1}{2}\, x_2]$. By computing the Hankel matrix

$$H_\sigma^{[1,x_1,x_2],\boldsymbol{u}} = \begin{bmatrix} \langle\sigma|\boldsymbol{u}_1\rangle & \langle\sigma|\boldsymbol{u}_2\rangle & \langle\sigma|\boldsymbol{u}_3\rangle \\ \langle\sigma|x_1\boldsymbol{u}_1\rangle & \langle\sigma|x_1\boldsymbol{u}_2\rangle & \langle\sigma|x_1\boldsymbol{u}_3\rangle \\ \langle\sigma|x_2\boldsymbol{u}_1\rangle & \langle\sigma|x_2\boldsymbol{u}_2\rangle & \langle\sigma|x_2\boldsymbol{u}_3\rangle \end{bmatrix} = \begin{bmatrix} 2 & 3 & -1 \\ 2\times1 & 3\times2 & -1\times3 \\ 2\times1 & 3\times2 & -1\times1 \end{bmatrix}$$

we deduce the weights $2, 3, -1$ and the frequencies $(1,1), (2,2), (3,1)$, which corresponds to the decomposition $\sigma = e^{y_1+y_2} + 3e^{2y_1+2y_2} - e^{2y_1+y_2}$ and $h(u_1, u_2) = 2 + 3 \cdot 2^{u_1+u_2} - 3^{u_1}$.

4.5.2 Example

Consider the following symmetric tensor of order $d = 4$, that is in the vector space $\mathbb{K}[x_0, x_1, x_2]_{[4]}$ of homogeneous polynomials of degree d :

$$\begin{aligned} \psi \;=\; & -{x_0}^4 - 24\, {x_0}^3 x_2 - 8\, {x_0}^3 x_1 - 60\, {x_0}^2 {x_2}^2 - 168\, {x_0}^2 x_1 x_2 - 12\, {x_0}^2 {x_1}^2 \\ & -96\, x_0 {x_2}^3 - 240\, x_0 x_1 {x_2}^2 - 384\, x_0 {x_1}^2 x_2 + 16\, x_0 {x_1}^3 \\ & -46\, {x_2}^4 - 200\, x_1 {x_2}^3 - 228\, {x_1}^2 {x_2}^2 - 296\, {x_1}^3 x_2 + 34\, {x_1}^4 \end{aligned}$$

By apolarity (see Section 1.2.2), we associate to ψ the dual element $\psi^* : p \mapsto \langle \psi, p \rangle_d \in \mathbb{K}[\boldsymbol{x}]_{[d]}{}^\star$. The Hankel matrix associated to ψ^* in degree 2, 2 for the set $B = \{1, x_1, x_2, x_1^2, x_1 x_2, x_2^2\}$

indexing the rows and columns is

$$H_{\psi^*}^{2,2} := \begin{bmatrix} -1 & -2 & -6 & -2 & -14 & -10 \\ -2 & -2 & -14 & 4 & -32 & -20 \\ -6 & -14 & -10 & -32 & -20 & -24 \\ -2 & 4 & -32 & 34 & -74 & -38 \\ -14 & -32 & -20 & -74 & -38 & -50 \\ -10 & -20 & -24 & -38 & -50 & -46 \end{bmatrix}$$

For $B = \{1, x_1, x_2\}$,

$$H_{\psi^*}^{B,B} = \begin{bmatrix} -1 & -2 & -6 \\ -2 & -2 & -14 \\ -6 & -14 & -10 \end{bmatrix} \qquad H_{\psi^*}^{B,x_1B} = \begin{bmatrix} -2 & -2 & -14 \\ -2 & 4 & -32 \\ -14 & -32 & -20 \end{bmatrix} \qquad H_{\psi^*}^{B,x_2B} = \begin{bmatrix} -6 & -14 & -10 \\ -14 & -32 & -20 \\ -10 & -20 & -24 \end{bmatrix}$$

The matrix of multiplication by x_2 in $B = \{1, x_1, x_2\}$ is

$$M_2 = (H_{\psi^*}^{B,B})^{-1} H_{\psi^*}^{B,x_2B} = \begin{bmatrix} 0 & -2 & -2 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 1 & \frac{5}{2} & \frac{3}{2} \end{bmatrix}.$$

Its eigenvalues are $[-1, 1, 2]$ and the eigenvectors:

$$U := \begin{bmatrix} 0 & 2 & -1 \\ \frac{1}{4} & -\frac{3}{4} & \frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

that is the polynomials $U(x) = \left[\ \frac{1}{4}x_1 - \frac{1}{4}x_2 \quad 2 - \frac{3}{4}x_1 - \frac{1}{4}x_2 \quad -1 + \frac{1}{2}x_1 + \frac{1}{2}x_2\ \right]$. We deduce the weights and the frequencies:

$$H_{\psi^*}^{[1,x_1,x_2],U} = \begin{bmatrix} 1 & 1 & -3 \\ 1 \times 3 & 1 \times 1 & -3 \times 2 \\ 1 \times -1 & 1 \times 1 & -3 \times 2 \end{bmatrix}.$$

This gives the weights $1, 1, -3$ and the frequency points $(3, -1), (1, 1), (2, 2)$ corresponding to the decomposition

$$\psi^*(y) = \mathfrak{e}_{(3,-1)}(y) + \mathfrak{e}_{(1,1)} - 3\,\mathfrak{e}_{(2,2)}(y) + \mathcal{O}(y)^4$$

and the tensor decomposition

$$\psi = (x_0 + 3\,x_1 - x_2)^4 + (x_0 + x_1 + x_2)^4 - 3\,(x_0 + 2\,x_2 + 2\,x_2)^4$$

4.6 Border basis, orthogonal polynomials

An important step in the decomposition method consists in computing a basis B of $\mathscr{A}_\sigma$. In this section, we describe how to compute a monomial basis $B = \{\mathbf{x}^\beta\}$ and two other bases $\mathbf{p} = (p_\beta)$ and $\mathbf{q} = (q_\beta)$, which are pairwise orthogonal for the inner product $\langle \cdot, \cdot \rangle_\sigma$:

$$\langle p_\beta, q_{\beta'} \rangle_\sigma = \begin{cases} 1 & \text{if } \beta = \beta' \\ 0 & \text{otherwise}. \end{cases}$$

Such pairwise orthogonal bases of $\mathscr{A}_\sigma$ exist, since $\mathscr{A}_\sigma$ is an Artinian Gorenstein algebra and $\langle \cdot, \cdot \rangle_\sigma$ is non-degenerate (Proposition 4.3.2).

To compute these pairwise orthogonal bases, we will use a projection process, similar to Gram-Schmidt orthogonalization process. The main difference is that we compute pairs p_β, q_β of orthogonal polynomials. As the inner product $\langle \cdot, \cdot \rangle_\sigma$ may be isotropic, the two polynomials p_β, q_β may not be equal, up to a scalar.

The method proceeds inductively starting from $\mathbf{b} = [\,]$, extending the monomials basis $\mathbf{b}$ with new monomials $\mathbf{x}^\alpha$, projecting them onto the space spanned by $\mathbf{b}$:

$$p_\alpha = \mathbf{x}^\alpha - \sum_{\beta \in \mathbf{b}} \langle \mathbf{x}^\alpha, q_\beta \rangle_\sigma p_\beta$$

and computing q_α, if it exists, such that $\langle p_\alpha, q_\alpha \rangle_\sigma = 1$ and $\langle \mathbf{x}^\beta, q_\alpha \rangle_\sigma = 0$ for $\beta \in \mathbf{b}$. Here is a more detailled description of the algorithm:

Algorithm 4.6.1: Orthogonal bases

Input: the coefficients σ_α of a series $\sigma \in \mathbb{K}[[\boldsymbol{y}]]$ for $\alpha \in \mathbf{a} \subset \mathbb{N}^n$.

Let $\mathbf{b} := []$; $\mathbf{b}' := []$; $\mathbf{d} = []$; $\mathbf{n} := [\mathbf{0}]$; $\mathbf{s} := \mathbf{a}$; $\mathbf{s}' := \mathbf{a}$; $l := 0$;

while $\mathbf{n} \neq \emptyset$ **do**

$l := l + 1$;

for ***each*** $\alpha \in \mathbf{n}$ **do**

a) compute $p_\alpha = \mathbf{x}^\alpha - \sum_{\beta \in B} \langle \mathbf{x}^\alpha, q_\beta \rangle_\sigma p_\beta$;

b) find the first $\alpha' \in \mathbf{s}'$ such that $\mathbf{x}^{\alpha'} p_\alpha \in \langle \mathbf{a} \rangle$ and $\langle \mathbf{x}^{\alpha'}, p_\alpha \rangle_\sigma \neq 0$;

c) **if** *such an* α' *exists* **then**

let $q_\alpha := \frac{1}{\langle \mathbf{x}^{\alpha'}, p_\alpha \rangle_\sigma} \left(\mathbf{x}^{\alpha'} - \sum_{\beta \in B} \langle \mathbf{x}^{\alpha'}, p_\beta \rangle_\sigma q_\beta \right)$;

add α to $\mathbf{b}$; remove α from $\mathbf{s}$;

add α' to $\mathbf{b}'$; remove α' from $\mathbf{s}'$;

else

add α to $\mathbf{d}$;

end

$\boldsymbol{n} := \text{next}(\boldsymbol{b}, \boldsymbol{d}, \boldsymbol{s})$;

end

Output:

- monomial sets $\mathbf{b} = [\beta_1, \ldots, \beta_r] \subset \mathbf{a}$, $\mathbf{b}' = [\beta_1', \ldots, \beta_r'] \subset \mathbf{a}$.
- pairwise orthogonal bases $\mathbf{p} = (p_{\beta_i})$, $\mathbf{q} = (q_{\beta_i})$ for $\langle \cdot, \cdot \rangle_\sigma$.
- the relations $p_\alpha := \mathbf{x}^\alpha - \sum_{i=1}^r \langle \mathbf{x}^\alpha, q_{\beta_i} \rangle_\sigma p_{\beta_i}$ for $\alpha \in \mathbf{d}$.

The algorithm manipulates the ordered lists $\mathbf{b}, \mathbf{d}, \mathbf{s}, \mathbf{s}'$ of exponents, identified with monomials. The monomials are ordered according to a total order denoted $\prec$. The index l is the loop index.

The algorithm uses the function next($\mathbf{b}, \mathbf{d}, \mathbf{s}$), which computes the set of monomials α in $\partial \mathbf{b} \cap \mathbf{s}$, which are not in $\mathbf{d}$ and such that $\alpha + \mathbf{b}' \subset \mathbf{a}$.

We verify that at each loop of the algorithm, the lists $\mathbf{b}$ and $\mathbf{s}$ (resp. $\mathbf{b}'$ and $\mathbf{s}'$) are disjoint and $\mathbf{b} \cup \mathbf{s} = \mathbf{a}$ (resp. $\mathbf{b}' \cup \mathbf{s}' = \mathbf{a}$).

We also verify by induction that at each loop, $\langle \mathbf{x}^{\mathbf{b}} \rangle = \langle p_\beta \mid \beta \in \mathbf{b} \rangle$ and $\langle \mathbf{x}^{\mathbf{b}'} \rangle = \langle q_\beta \mid \beta \in \mathbf{b} \rangle$.

The following properties are also satisfied at the end of the algorithm:

Theorem 4.6.1 *Let $\mathbf{b} = [\beta_1, \ldots, \beta_r]$, $\mathbf{b}' = [\beta_1', \ldots, \beta_r']$, $\mathbf{p} = [p_{\beta_1}, \ldots, p_{\beta_r}]$, $\mathbf{q} = [q_{\beta_1}, \ldots, q_{\beta_r}]$ be the output of Algorithm 4.6.1. Let $V = \langle \mathbf{x}^{\mathbf{b}^+} \rangle$. If there exists a vector space V' connected to 1 such that $\mathbf{x}^{(\mathbf{b}')^+} \subset V'$ and $V \cdot V' = \langle \mathbf{x}^{\mathbf{a}} \rangle$. Then σ coincides on $\langle \mathbf{x}^{\mathbf{a}} \rangle$ with the unique series $\tilde{\sigma} \in \mathbb{K}[[\mathbf{y}]]$ such that $\tilde{\sigma}_{|\langle \mathbf{x}^{\mathbf{a}} \rangle} = \sigma$ and $\operatorname{rank} H_{\tilde{\sigma}} = r$ and we have the following properties:*

- *$(\mathbf{p}, \mathbf{q})$ are pairwise orthogonal bases of $\mathscr{A}_{\tilde{\sigma}}$ for the inner product $\langle \cdot, \cdot \rangle_{\tilde{\sigma}}$.*
- *The family $\left\{p_\alpha = \mathbf{x}^\alpha - \sum_{i=1}^{r} \langle \mathbf{x}^\alpha, q_{\beta_i} \rangle_\sigma p_{\beta_i}, \alpha \in \mathbf{d}\right\}$ is a border basis of the ideal $I_{\tilde{\sigma}}$, with respect to $\mathbf{x}^{\mathbf{b}}$.*
- *The matrix of multiplication by x_k in the basis $\mathbf{p}$ (resp. $\mathbf{q}$) of $\mathscr{A}_{\tilde{\sigma}}$ is $M_k := (\langle \sigma | x_k p_{\beta_j} q_{\beta_i} \rangle)_{1 \leqslant i,j \leqslant r}$ (resp. M_k^t).*

Proof. By construction, $V = \langle \mathbf{x}^{\mathbf{b}^+} \rangle$ is connected to 1 and $\mathbf{x}^{\mathbf{b}}$ contains 1, otherwise $\sigma = 0$. As V' contains $\mathbf{x}^{\mathbf{b}'}$ and $V \cdot V' = \langle \mathbf{x}^{\mathbf{a}} \rangle$, we have $\forall \alpha \in \partial \mathbf{b}, \mathbf{x}^\alpha \cdot \mathbf{x}^{\mathbf{b}'} \subset \mathbf{x}^{\mathbf{a}}$. Thus when the algorithm stops, we have $\mathbf{n} = \emptyset$ and $\partial \mathbf{b} = \mathbf{d}$. By construction, for $\alpha \in \mathbf{d}$ the polynomials $p_\alpha = \mathbf{x}^\alpha - \sum_{\beta \in \mathbf{b}} \langle \mathbf{x}^\alpha, q_\beta \rangle_\sigma p_\beta$ are orthogonal to $\langle q_\beta \mid \beta \in \mathbf{b} \rangle = \langle \mathbf{x}^{\mathbf{b}'} \rangle$. As $\alpha \in \mathbf{d}$, for each $v' \in V'$, we have moreover $\langle p_\alpha, v' \rangle_\sigma = 0$.

A basis of V is formed by the polynomials p_α for $\alpha \in \mathbf{b}^+$ since $\langle p_\beta \mid \beta \in \mathbf{b} \rangle = \langle \mathbf{x}^{\mathbf{b}} \rangle$ and $p_\alpha = \mathbf{x}^\alpha + b_\alpha$ with $b_\alpha \in \langle \mathbf{x}^{\mathbf{b}} \rangle$ for $\alpha \in \mathbf{d} = \partial \mathbf{b}$. The matrix of $H_\sigma^{V,V'}$ in this basis of V and in a basis of V', which first elements are $q_{\beta_1}, \ldots, q_{\beta_r}$, is of the form

$$H_\sigma^{V,V'} = \begin{pmatrix} \mathbb{I}_r & 0 \\ * & 0 \end{pmatrix}$$

where $\mathbb{I}_r$ is the identity matrix of size r. The kernel of $H_\sigma^{V,V'}$ is generated by the polynomials p_α for $\alpha \in \mathbf{d}$.

By Theorem 4.3.4, σ coincides on $V \cdot V' = \langle \mathbf{x}^{\mathbf{a}} \rangle$ with a series $\tilde{\sigma}$ such that $\mathbf{x}^{\mathbf{b}}$ is a basis of $\mathscr{A}_{\tilde{\sigma}} = \mathbb{K}[\mathbf{x}]/I_{\tilde{\sigma}}$ and $I_{\tilde{\sigma}} = (\ker H_{\tilde{\sigma}}^{V,V'}) = (p_\alpha)_{\alpha \in \mathbf{d}}$.

As $p_\alpha = \mathbf{x}^\alpha + b_\alpha$ with $\alpha \in \partial \mathbf{b}$ and $b_\alpha \in \langle \mathbf{x}^{\mathbf{b}} \rangle$, $(p_\alpha)_{\alpha \in \partial \mathbf{b}}$ is a border basis with respect to $\mathbf{x}^{\mathbf{b}}$ for the ideal $I_{\tilde{\sigma}}$, since $\mathbf{x}^{\mathbf{b}}$ is a basis of of $\mathscr{A}_{\tilde{\sigma}}$.

This shows that $\operatorname{rank} H_{\tilde{\sigma}} = \dim \mathscr{A}_{\tilde{\sigma}} = |\mathbf{b}| = r$. By construction, $(\mathbf{p}, \mathbf{q})$ are pairwise orthogonal for the inner product $\langle \cdot, \cdot \rangle_\sigma$, which coincides with $\langle \cdot, \cdot \rangle_{\tilde{\sigma}}$ on $\langle \mathbf{x}^{\mathbf{a}} \rangle$. Thus they are pairwise orthogonal bases of $\mathscr{A}_{\tilde{\sigma}}$ for the inner product $\langle \cdot, \cdot \rangle_{\tilde{\sigma}}$.

As we have $x_k p_{\beta_j} \equiv \sum_{i=1}^{r} \langle x_k p_{\beta_j}, q_{\beta_i} \rangle_\sigma p_{\beta_i}$, the matrix of multiplication by x_k in the basis $\mathbf{p}$ of $\mathscr{A}_{\tilde{\sigma}}$ is $M_k := (\langle x_k p_{\beta_j}, q_{\beta_i} \rangle_\sigma)_{1 \leqslant i,j \leqslant r} = (\langle \sigma | x_k p_{\beta_j} q_{\beta_i} \rangle)_{1 \leqslant i,j \leqslant r}$. Exchanging the role of $\mathbf{p}$ and $\mathbf{q}$, we obtain M_k^t for the matrix of multiplication by x_k in the basis $\boldsymbol{q}$. □

Remark 4.6.2 *If the polynomials p_α, q_α are at most of degree d, then only the coefficients of σ of degree $\leqslant 2d + 1$ are involved in this computation. In this case, the border basis and the decomposition of the series σ as a sum of exponential polynomials can be computed from these first coefficients.*

Remark 4.6.3 *When the monomials in* $\mathbf{s}$ *are chosen according to a monomial ordering* $\prec$, *the polynomials* $p_\alpha = \mathbf{x}^\alpha + b_\alpha$, $\alpha \in \mathbf{d}$ *are constructed in such a way that their leading term is* $\mathbf{x}^\alpha$. *They form a Gröbner basis of the ideal* $I_{\tilde{\sigma}}$. *To construct a minimal Gröbner basis of* $I_{\tilde{\sigma}}$ *for the monomial ordering* $\prec$, *it suffices to keep the elements* p_α *with* $\alpha \in \mathbf{d}$ *minimal for the division.*

Remark 4.6.4 *The computation can be simplified, when* $\langle\cdot,\cdot\rangle_\sigma$ *is semi-definite, that is, when for all* $p \in \langle \mathbf{x}^a \rangle$ *such that* $p^2 \in \langle \mathbf{x}^{\mathbf{a}} \rangle$, *we have* $\langle p, p\rangle_\sigma = 0$ *implies that* $\forall \alpha \in \mathbf{a}$ *with* $\mathbf{x}^\alpha p \in \langle \mathbf{x}^{\mathbf{a}} \rangle$, $\langle p, \mathbf{x}^\alpha \rangle_\sigma = 0$. *In this case, the algorithm constructs a family of orthogonal polynomials* $\mathbf{p} = [p_{\beta_1}, \ldots, p_{\beta_r}]$ *and* $\mathbf{q} = [q_{\beta_1}, \ldots, q_{\beta_r}]$ *with* $q_{\beta_i} = \frac{1}{\langle p_{\beta_i}, p_{\beta i}\rangle_\sigma} p_{\beta_i}$ *and we have* $\mathbf{b} = \mathbf{b}'$. *Indeed, in the while loop for each* $\alpha \in \mathbf{n}$, *either* $\langle p_\alpha, p_\alpha \rangle_\sigma = 0$, *which implies that* $\forall \alpha' \in \mathbf{t}$ *with* $\mathbf{x}^{\alpha'} p_\alpha \in \langle \mathbf{x}^{\mathbf{a}} \rangle$, $\langle \mathbf{x}^{\alpha'}, p_\alpha \rangle_\sigma = 0$, *so that* $\alpha \in \mathbf{d}$, *or* $\langle p_\alpha, p_\alpha \rangle_\sigma = \langle \mathbf{x}^\alpha, p_\alpha \rangle_\sigma \neq 0$ *and the first* $\alpha' \in \mathbf{t}$ *such that* $\langle \mathbf{x}^{\alpha'}, p_\alpha \rangle_\sigma$ *is* $\alpha' = \alpha \in \mathbf{b}$.

If $\mathbb{K} = \mathbb{R}$ *and* σ *is semi-definite positive, then the polynomials* $\frac{1}{\sqrt{\langle p_{\beta_i}, p_{\beta i}\rangle_\sigma}} p_{\beta_i}$ *are classical orthogonal polynomials for* $\langle\cdot,\cdot\rangle_\sigma$.

We can now describe the decomposition algorithm of polynomial-exponential series, obtained by combining the algorithm for computing bases of $\mathscr{A}_\sigma$ and the algorithm for computing the frequency points and the weights:

Algorithm 4.6.2: Polynomial-Exponential decomposition

Input: the coefficients σ_α of a series $\sigma \in \mathbb{K}[[\boldsymbol{y}]]$ for $\alpha \in \mathbf{a} \subset \mathbb{N}^n$.

- Apply Algorithm 4.6.1 to compute bases $B = \boldsymbol{x}^{\boldsymbol{b}}$, $B' = \boldsymbol{x}^{\boldsymbol{b}'}$ of $\mathscr{A}_\sigma$;
- **if** $\exists V' \supset B'$ *s.t.* $\langle V' \cdot B^+ \rangle = \langle \mathbf{x}^{\mathbf{a}} \rangle$ **then**
  Apply Algorithm 4.5.1.

Output: the polynomial-exponential series $\sum_{i=1}^{r} \omega_i(\boldsymbol{y}) \mathfrak{e}_{\xi_i}(\boldsymbol{y})$ with $\omega_i(\boldsymbol{y}) \in \mathbb{K}[\boldsymbol{y}]$, $\xi_i \in \mathbb{K}^n$ with the same Taylor coefficients σ_α as σ for $\alpha \in \mathbf{a} \subset \mathbb{N}^n$.

4.6.1 Examples

Example 4.6.5 *Let* $n = 1$ *and* $\sigma(y) = \frac{y^d}{d!} \in \mathbb{K}[[y]]$ *with* $0 < d$ *and* $a \neq 0 \in \mathbb{K}$.

In the first step of the algorithm, we take $p_1 = 1$ *and compute the first* i *such that* $\langle x^i, p_1 \rangle_\sigma$ *is not zero. This yields* $\mathbf{b} = [1]$, $\mathbf{b}' = [x^d]$ *and* $q_1 = x^d$.

In a second step, we have $p_x = x - \langle x, q_1 \rangle_\sigma p_1 = x$. *The first* i *such that* $\langle x^i, p_x \rangle_\sigma$ *is not zero yields* $\mathbf{b} = [1, x]$, $\mathbf{b}' = [x^d, x^{d-1}]$ *and* $q_x = x^{d-1} - \langle x^{d-1}, p_1 \rangle_\sigma q_1 = x^{d-1}$.

We repeat this computation until $\mathbf{b} = [1, \ldots, x^d]$, $\mathbf{b}' = [x^d, x^{d-1}, \ldots, 1]$ *with* $p_{x^i} = x^i$, $q_{x^i} = x^{d-i}$ *for* $i = 0, \ldots, d$.

In the following step, we have $p_{x^{d+1}} = x^{d+1} - \langle x^{d+1}, q_1 \rangle_\sigma p_1 - \cdots - \langle x^{d+1}, q_{x^d} \rangle_\sigma p_{x^d} = x^{d+1}$. *The algorithm stops and outputs* $\mathbf{b} = [1, \ldots, x^d]$, $\mathbf{b}' = [x^d, x^{d-1}, \ldots, 1]$, $p_{x^{d+1}} = x^{d+1}$.

Example 4.6.6 *We consider the function* $h(u_1,u_2) = 2+3 \cdot 2^{u_1}2^{u_2}-3^{u_1}$. *Its associated generating series is* $\sigma = \sum_{\alpha\in\mathbb{N}^2} h(\alpha)\frac{y^\alpha}{\alpha!}$. *Its (truncated) moment matrix is*

$$H_\sigma^{[1,x_1,x_2,x_1^2,x_1x_2,x_2^2]} = \begin{bmatrix} h(0,0) & h(1,0) & h(0,1) & \cdots \\ h(1,0) & h(2,0) & h(1,1) & \cdots \\ h(0,1) & h(1,1) & h(0,2) & \cdots \\ \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \end{bmatrix} = \begin{bmatrix} 4 & 5 & 7 & 5 & 11 & 13 \\ 5 & 5 & 11 & -1 & 17 & 23 \\ 7 & 11 & 13 & 17 & 23 & 25 \\ 5 & -1 & 17 & -31 & 23 & 41 \\ 11 & 17 & 23 & 23 & 41 & 47 \\ 13 & 23 & 25 & 41 & 47 & 49 \end{bmatrix}.$$

At the first step, we have $\mathbf{b} = [1]$, $\mathbf{p} = [1]$, $\mathbf{q} = [\frac{1}{4}]$. *At the second step, we compute* $\mathbf{b} = [1, x_1, x_2]$, $\mathbf{p} = [1, x_1 - \frac{5}{4}, x_2 + \frac{9}{5}x_1 - 4] = [p_1, p_{x_1}, p_{x_2}]$ *and* $\mathbf{q} = [\frac{1}{4}p_1, -\frac{4}{5}p_{x_1}, \frac{5}{24}p_{x_2}]$. *At the third step,* $\mathbf{d} = [x_1^2, x_1x_2, x_2^2]$ *and the algorithm stops. We obtain the following generators of* $\ker H_\sigma$*:*

$$\begin{aligned} p_{x_1^2} &= x_1^2 + x_2 - 4x_1 + 2 \\ p_{x_1x_2} &= x_1x_2 - 2x_2 - x_1 + 2 \\ p_{x_2^2} &= x_2^2 - 3x_2 + 2 \end{aligned}$$

We have modulo $\ker H_\sigma$*:*

$$\begin{aligned} x_1\boldsymbol{p}_1 &\equiv \sum_i \langle x_1\boldsymbol{p}_1, \boldsymbol{q}_i\rangle_\sigma \boldsymbol{p}_i = \frac{5}{4}\boldsymbol{p}_1 + \boldsymbol{p}_2 \\ x_1\boldsymbol{p}_2 &\equiv \sum_i \langle x_1\boldsymbol{p}_2, \boldsymbol{q}_i\rangle_\sigma \boldsymbol{p}_i = -\frac{5}{16}\boldsymbol{p}_1 + \frac{91}{20}\boldsymbol{p}_2 - \boldsymbol{p}_3 \\ x_1\boldsymbol{p}_3 &\equiv \sum_i \langle x_1\boldsymbol{p}_3, \boldsymbol{q}_i\rangle_\sigma \boldsymbol{p}_i = \frac{96}{25}\boldsymbol{p}_2 + \frac{1}{5}\boldsymbol{p}_3 \end{aligned}$$

The matrix of multiplication by x_1 *in the basis* $\mathbf{p}$ *is*

$$M_1 = \begin{bmatrix} \frac{5}{4} & -\frac{5}{16} & 0 \\ 1 & \frac{91}{20} & \frac{96}{25} \\ 0 & -1 & \frac{1}{5} \end{bmatrix}$$

Its eigenvalues are $[1,2,3]$ *and the corresponding matrix of eigenvectors is*

$$U := \begin{bmatrix} \frac{1}{2} & \frac{3}{4} & -\frac{1}{4} \\ \frac{2}{5} & -\frac{9}{5} & \frac{7}{5} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix},$$

that is, the polynomials $U(x) = [2 - \frac{1}{2}\,x_1 - \frac{1}{2}\,x_2, -1 + x_2, \frac{1}{2}\,x_1 - \frac{1}{2}\,x_2]$. *By computing the Hankel matrix*

$$H_\sigma^{U,[1,x_1,x_2]} = \begin{bmatrix} 2 & 3 & -1 \\ 2\times 1 & 3\times 2 & -1\times 3 \\ 2\times 1 & 3\times 2 & -1\times 1 \end{bmatrix}$$

we deduce the weights $2, 3, -1$ *and the frequencies* $(1,1), (2,2), (3,1)$*, which corresponds to the decomposition* $\sigma = e^{y_1+y_2} + 3e^{2y_1+2y_2} - e^{2y_1+y_2}$ *associated to* $h(u_1, u_2) = 2 + 3 \cdot 2^{u_1+u_2} - 3^{u_1}$.

4.7 Structured low rank decomposition of Hankel operators

In this section, $m = 1$ and we consider Hankel operators associated to symbols $\sigma \in \mathbb{K}[\boldsymbol{x}]^{\star}$.

4.7.1 Simple roots

Let $\sigma = \sum_{i=1}^{r} \omega_i \mathfrak{e}_{\xi_i}$ with $\omega_i \in \mathbb{K} \setminus \{0\}$. Let us recall other relations between the structured matrices involved in this decomposition problem, that are useful to analyse the numerical behavior of the method. For more details, see e.g. [MP00]. Such decompositions, referred as Carathéodory-Fejér-Pisarenko decompositions in [YXS15]. They can be used to recover the decomposition of the series in Pencil-like methods.

Definition 4.7.1 *Let* $B = \{b_1, \ldots, b_r\}$ *be a family of polynomials. We define the B-Vandermonde matrix of the points* $\xi_1, \ldots, \xi_r \in \mathbb{C}^n$ *as*

$$V_{B,\xi} = (\langle \mathfrak{e}_{\xi_j} | b_i \rangle)_{1 \le i,j \le r} = (b_i(\xi_j))_{1 \le i,j \le r}.$$

By remark 3.3.3, if $\{\mathfrak{e}_{\xi_1}, \ldots, \mathfrak{e}_{\xi_r}\}$ is a basis of $\mathscr{A}_\sigma^*$ and B is a basis of $\mathscr{A}_\sigma$, then $V_{B,\xi}$ is the matrix of coefficients of $\mathfrak{e}_{\xi_1}, \ldots, \mathfrak{e}_{\xi_r}$ in the dual basis of B in $\mathscr{A}_\sigma^*$ and it is invertible. Conversely, if $\{\mathfrak{e}_{\xi_1}, \ldots, \mathfrak{e}_{\xi_r}\}$ is a basis of $\mathscr{A}_\sigma^*$, we check that $V_{B,\xi}$ is invertible and that $B = \{b_1, \ldots, b_r\}$ is a basis of $\mathscr{A}_\sigma$.

Proposition 4.7.2 *Suppose that* $\sigma = \sum_{k=1}^{r} \omega_k \mathfrak{e}_{\xi_k}(\boldsymbol{y})$ *with* $\xi_1, \ldots, \xi_r \in \mathbb{K}^n$ *pairwise distinct and* $\omega_1, \ldots, \omega_r \in \mathbb{K} \setminus \{0\}$*. Let* $D_{\boldsymbol{\omega}} = \mathrm{diag}(\omega_1, \ldots, \omega_r)$ *be the diagonal matrix associated to the weights* ω_i *and for* $g \in \mathbb{K}[\boldsymbol{x}]$*, let* $D_g = \mathrm{diag}(g(\xi_1), \ldots, g(\xi_r))$ *be the diagonal matrices associated to* $g(\xi_1), \ldots, g(\xi_r)$*. For any family* B, B' *of* $\mathbb{K}[\boldsymbol{x}]$*, we have*

$$\begin{array}{rcl} H_\sigma^{B,B'} & = & V_{B',\xi} D_{\boldsymbol{\omega}} V_{B,\xi}^t \\ H_{g\star\sigma}^{B,B'} & = & V_{B',\xi} D_{\boldsymbol{\omega}} D_g V_{B,\xi}^t = V_{B',\xi} D_g D_{\boldsymbol{\omega}} V_{B,\xi}^t \end{array}$$

If moreover B *is a basis of* $\mathscr{A}_\sigma$*, then* $V_{B,\xi}$ *is invertible and*

$$(M_g^B)^t \quad = \quad V_{B,\xi} D_g V_{B,\xi}^{-1}$$

Proof. If $\sigma = \sum_{k=1}^{r} \omega_k \mathfrak{e}_{\xi_k}(\boldsymbol{y})$ and $B = \{b_1, \ldots, b_r\}, B' = \{b'_1, \ldots, b'_r\}$ are bases of $\mathscr{A}_\sigma$, then

$$H_\sigma^{B,B'} = \left[\sum_{k=1}^{r} \omega_k b'_i(\xi_k) b_j(\xi_k) \right]_{i,j=1,\ldots,r} = V_{B',\xi} D_{\boldsymbol{\omega}} V_{B,\xi}^t.$$

By a similar explicit computation, we check that $H^{B,B'}_{g\star\sigma} = V_{B',\xi} D_{\boldsymbol{\omega}} D_g V^t_{B,\xi}$. Equation (4.3) implies that $(M^B_g)^t = H^{B,B}_{g\star\sigma}(H^{B,B}_{\sigma})^{-1} = V_{B,\xi} D_g V^{-1}_{B,\xi}$. □

4.7.2 Multiple roots

The relations between Vandermonde matrices and Hankel matrices (Proposition 4.7.2) can be generalized to the case of multiple roots. Let $\sigma = \sum_{k=1}^{r'} \omega_k(\mathbf{y}) \mathfrak{e}_{\xi_k}(\mathbf{y})$ with $\xi_1, \ldots, \xi_{r'} \in \mathbb{K}^n$ pairwise distinct, $\omega_1(\mathbf{y}), \ldots, \omega_{r'}(\mathbf{y}) \in \mathbb{K}[\mathbf{y}] \setminus \{0\}$. To deduce a decomposition of $H^{B,B'}_{\sigma}$ similar to the decomposition of Proposition 4.7.2 for multiple roots, we introduce the Wronskian of a set $B = \{b_1, \ldots, b_l\} \subset \mathbb{K}[\mathbf{x}]$ and a set of exponents $\Gamma = \{\gamma_1, \ldots, \gamma_s\} \subset \mathbb{N}^n$ at a point $\xi \in \mathbb{K}^n$:

$$W_{B,\Gamma,\xi} = \left[\frac{1}{\gamma_j!} \partial^{\gamma_j}(b_i)(\xi) \right]_{1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s}.$$

For a collection $\boldsymbol{\Gamma} = \{\Gamma_1, \ldots, \Gamma_{r'}\}$ with $\Gamma_1, \ldots, \Gamma_{r'} \subset \mathbb{N}^n$ and points $\boldsymbol{\xi} = \{\xi_1, \ldots, \xi_{r'}\} \subset \mathbb{K}^n$ let

$$W_{B,\boldsymbol{\Gamma},\boldsymbol{\xi}} = [W_{B,\Gamma_1,\xi_1}, \ldots, W_{B,\Gamma_{r'},\xi_{r'}}]$$

be the matrix obtained by concatenation of the columns of W_{B,Γ_k,ξ_k}, $k = 1, \ldots, r'$.

We consider the monomial decomposition $\omega_k(\mathbf{y}) = \sum_{\alpha \in A_k} \omega_{k,\alpha} (\mathbf{x} - \xi_k)^\alpha$ with $\omega_{k,\alpha} \neq 0$. We denote by Γ_k the set of all the exponents $\alpha \in A_k$ in this decomposition and all their divisors $\beta = (\beta_1, \ldots, \beta_n)$ with $\beta \ll \alpha$. Let us denote by $\gamma_1, \ldots, \gamma_{s_k}$ the elements of Γ_k.

Let $\Delta^{\Gamma_k}_{\omega_k} = [(\gamma_i + \gamma_j)! \omega_{k,\gamma_i+\gamma_j}]_{1 \leqslant i,j \leqslant s_k}$ with the convention that $\omega_{k,\gamma_i+\gamma_j} = 0$ if $\gamma_i + \gamma_j \notin A_k$ is not a monomial exponent of $\omega_k(\mathbf{y})$. Let $\Delta^{\boldsymbol{\Gamma}}_{\boldsymbol{\omega}}$ be the block diagonal matrix, which diagonal blocks are $\Delta^{\Gamma_k}_{\omega_k}$, $k = 1, \ldots, r'$.

The following decomposition generalizes the Carathéodory-Fejér decomposition in the case of multiple roots (it is also implied by rank deficiency conditions):

Proposition 4.7.3 *Suppose that* $\sigma = \sum_{k=1}^{r'} \omega_k(\mathbf{y}) \mathfrak{e}_{\xi_k}(\mathbf{y})$ *with* $\xi_1, \ldots, \xi_{r'} \in \mathbb{K}^n$ *pairwise distinct,* $\omega_1(\mathbf{y}), \ldots, \omega_{r'}(\mathbf{y}) \in \mathbb{K}[\mathbf{y}] \setminus \{0\}$. *For* $g \in \mathbb{K}[\mathbf{x}]$, $g \circledast \boldsymbol{\omega} = [g(\xi_1 + \partial_y)(\omega_1), \ldots, g(\xi_{r'} + \partial_y)(\omega_{r'})]$. *For any set* $B, B' \subset \mathbb{K}[\mathbf{x}]$ *of size* l*, we have*

$$\begin{aligned} H^{B,B'}_{\sigma} &= W_{B',\boldsymbol{\Gamma},\xi} \Delta^{\boldsymbol{\Gamma}}_{\boldsymbol{\omega}} W^t_{B,\boldsymbol{\Gamma},\xi} \\ H^{B,B'}_{g\star\sigma} &= W_{B',\boldsymbol{\Gamma},\xi} \Delta^{\boldsymbol{\Gamma}}_{g\circledast\boldsymbol{\omega}} W^t_{B,\boldsymbol{\Gamma},\xi} \end{aligned}$$

If moreover B *is a basis of* $\mathscr{A}_\sigma$*, then* $W_{B',\boldsymbol{\Gamma},\xi}$ *and* $\Delta^{\boldsymbol{\Gamma}}_{\boldsymbol{\omega}}$ *are invertible and the matrix of multiplication by* g *in the basis* B *of* $\mathscr{A}_\sigma$ *is*

$$M^B_g = W^{-t}_{B,\boldsymbol{\Gamma},\xi} (\Delta^{\boldsymbol{\Gamma}}_{\boldsymbol{\omega}})^{-1} \Delta_{g\circledast\boldsymbol{\omega}} W^{-t}_{B,\boldsymbol{\Gamma},\xi}.$$

Proof. By the relation (2.3), we have

$$H_\sigma^{B,B'} = \left[\sum_{k=1}^{r'} \omega_k(\partial_{x_1},\ldots,\partial_{x_n})(b_i'b_j)(\xi_k)\right]_{1\leqslant i,j\leqslant l}.$$

By expansion, we obtain

$$\omega_k(\partial_{x_1},\ldots,\partial_{x_n})(b_i'b_j)(\xi_k) = \sum_{\alpha\in A_k} \omega_{k,\alpha}\partial_{\boldsymbol{x}}^{\alpha}(b_i'b_j)(\xi_k).$$

By Leibniz rule, we have

$$\partial_{\boldsymbol{x}}^{\alpha}(b_i'b_j) = \sum_{\beta\ll\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!}\partial_{\boldsymbol{x}}^{\beta}(b_i')\partial_{\boldsymbol{x}}^{\alpha-\beta}(b_j) = \alpha!\sum_{\beta\ll\alpha}\frac{\partial_{\boldsymbol{x}}^{\beta}(b_i')}{\beta!}\frac{\partial_{\boldsymbol{x}}^{\alpha-\beta}(b_j)}{(\alpha-\beta)!}.$$

We deduce that

$$\begin{aligned}
\omega_k(\partial_{x_1},\ldots,\partial_{x_n})(b_i'b_j)(\xi_k) &= \sum_{\alpha\in A_k}\omega_{k,\alpha}\partial_{\boldsymbol{x}}^{\alpha}(b_i'b_j)(\xi_k)\\
&= \sum_{\alpha\in A_k}\alpha!\,\omega_{k,\alpha}\sum_{\beta\ll\alpha}\frac{\partial_{\boldsymbol{x}}^{\beta}(b_i')}{\beta!}(\xi_k)\frac{\partial_{\boldsymbol{x}}^{\alpha-\beta}(b_j)}{(\alpha-\beta)!}(\xi_k)\\
&= W_{B',\Gamma_k,\xi_k}\Delta_{\omega_k}^{\Gamma_k}\quad W_{B,\Gamma_k,\xi_k}^{t}.
\end{aligned}$$

By concatenation of the columns of W_{B,Γ_k,ξ_k} and W_{B',Γ_k,ξ_k}, using the block diagonal matrix $\Delta_{\boldsymbol{\omega}}^{\Gamma}$, we obtain the decomposition of $H_\sigma^{B,B'} = W_{B',\Gamma_k,\xi_k}\Delta_{\omega_k}^{\Gamma_k}\quad W_{B,\Gamma_k,\xi_k}^{t}$.

By Lemma 2.2.5, we have

$$g\star\sigma = \sum_{k=1}^{r'} g(\xi_k+\partial_y)(\omega_k)\mathfrak{e}_{\xi_k} = \sum_{k=1}^{r'}(g\circledast\boldsymbol{\omega})_k\mathfrak{e}_{\xi_k}.$$

Thus, a similar computation yields the decomposition: $H_{g\star\sigma}^{B,B'} = W_{B',\Gamma,\xi}\Delta_{g\circledast\boldsymbol{\omega}}^{\Gamma}W_{B,\Gamma,\xi}^{t}$.

If B is a basis of $\mathscr{A}_\sigma$, then by Proposition 4.3.2, $H_\sigma^{B,B}$ is invertible, which implies that $W_{B',\Gamma,\xi}$ and $\Delta_{\boldsymbol{\omega}}^{\Gamma}$ are invertible. By Relation (4.3), we have

$$M_g^B = (H_\sigma^{B,B})^{-1}H_{g\star\sigma}^{B,B} = W_{B,\Gamma,\xi}^{-t}(\Delta_{\boldsymbol{\omega}}^{\Gamma})^{-1}\Delta_{g\circledast\boldsymbol{\omega}}W_{B,\Gamma,\xi}^{-t}.$$

□

4.8 Real positive series

In the case where all the coefficients of σ are in $\mathbb{R}$, we can consider the following property of positivity:

Definition 4.8.1 *An element $\sigma \in \mathbb{R}[[\boldsymbol{y}]] = \mathbb{R}[\boldsymbol{x}]^*$ is semi-definite positive if $\forall p \in \mathbb{R}[\boldsymbol{x}], \langle p, p\rangle_\sigma = \langle \sigma | p^2\rangle \geqslant 0$. It is denoted $\sigma \succcurlyeq 0$.*

The positivity of σ induces a nice property of its decomposition, which is an important ingredient of polynomial optimisation. It is saying that a positive measure on $\mathbb{R}^n$ with an Hankel operator of finite rank r is a convex combination of r distinct Dirac measures of $\mathbb{R}^n$. See e.g. [Lau09] for more details. For the sake of completeness, we give here a simple proof (see also [LLM+13][prop. 3.14]).

Proposition 4.8.2 *Let $\sigma \in \mathbb{R}[[\boldsymbol{y}]]$ of finite rank. Then $\sigma \succcurlyeq 0$, if and only if,*

$$\sigma = \sum_{i=1}^{r} \omega_i \, \mathfrak{e}_{\xi_i}$$

with $\omega_i > 0$, $\xi_i \in \mathbb{R}^n$.

Proof. If $\sigma = \sum_{i=1}^r \omega_i \, \mathfrak{e}_{\xi_i}$ with $\omega_i > 0$, $\xi_i \in \mathbb{R}^n$, then clearly $\forall p \in \mathbb{R}[\boldsymbol{x}]$,

$$\langle \sigma \mid p^2\rangle = \sum_{i=1}^{r} \omega_i \, p^2(\xi_i) \geqslant 0$$

and $\sigma \succcurlyeq 0$.

Conversely suppose that $\forall p \in \mathbb{R}[\boldsymbol{x}]$, $\langle \sigma \mid p^2\rangle \geqslant 0$. Then $p \in I_\sigma$, if and only if, $\langle \sigma \mid p^2\rangle = 0$. We check that I_σ is real radical: If $p^{2k} + \sum_j q_j^2 \in I_\sigma$ for some $k \in \mathbb{N}$, $p, q_j \in \mathbb{R}[\boldsymbol{x}]$ then

$$\langle \sigma \mid p^{2k} + \sum_j q_j^2\rangle = \langle \sigma \mid p^{2k}\rangle + \sum_j \langle \sigma \mid q_j^2\rangle = 0$$

which implies that $\langle \sigma \mid (p^k)^2\rangle = 0$, $\langle \sigma \mid q_j^2\rangle = 0$ and that $p^k, q_j \in I_\sigma$. Let $k' = \lceil \frac{k}{2} \rceil$. We have $\langle \sigma \mid (p^{k'})^2\rangle = 0$, which implies that $p^{k'} \in I_\sigma$. Iterating this reduction, we deduce that $p \in I_\sigma$. This shows that I_σ is real radical and $\mathscr{V}(I_\sigma) \subset \mathbb{R}^n$. By Proposition 4.3.3, we deduce that $\sigma = \sum_{i=1}^r \omega_i \, \mathfrak{e}_{\xi_i}$ with $\omega_i \in \mathbb{C} \setminus \{0\}$ and $\xi_i \in \mathbb{R}^n$. Let $\boldsymbol{u}_i \in \mathbb{R}[\boldsymbol{x}]$ be a family of interpolation polynomials at $\xi_i \in \mathbb{R}^n$: $\boldsymbol{u}_i(\xi_i) = 1$, $\boldsymbol{u}_i(\xi_j) = 0$ for $j \neq i$. Then $\langle \sigma \mid \boldsymbol{u}_i^2\rangle = \omega_i \in \mathbb{R}_+$. This proves that $\sigma(\boldsymbol{y}) = \sum_{i=1}^r \omega_i \, \mathfrak{e}_{\xi_i}(\boldsymbol{y})$ with $\omega_i > 0$, $\xi_i \in \mathbb{R}^n$. □

Chapter 5

Applications

5.1 Sparse decomposition from generating series

To exploit the previous results in the context of functional analysis or signal processing, we need to transform functions into series or sequences in $\mathbb{K}^{\mathbb{N}^n}$. Here is the general context that we consider, which extends the approach of [PP13] to multi-index sequences. We assume that $\mathbb{K}$ is algebraically close.

- Let $\mathscr{F}$ be a functional space (in which "leaves the functions, distributions or signals").
- Let $S_1, \ldots, S_n : \mathscr{F} \to \mathscr{F}$ be linear operators of $\mathscr{F}$, which are commuting: $S_i \circ S_j = S_j \circ S_i$.
- Let $\Delta : h \in \mathscr{F} \mapsto \Delta[h] \in \mathbb{K}$ be a linear functional on $\mathscr{F}$.

We associate to an element $h \in \mathscr{F}$, its generating series:

Definition 5.1.1 *For $h \in \mathscr{F}$, the generating series associated to h is*

$$\sigma_h(\mathbf{y}) \;=\; \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\Delta[S^\alpha(h)]\mathbf{y}^\alpha \tag{5.1}$$

where $S^\alpha = S_1^{\alpha_1} \circ \cdots \circ S_n^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$.

Definition 5.1.2 *We say that the* regularity condition *is satisfied if the map* $h \in \mathscr{F} \mapsto \sigma_h(\mathbf{y}) \in \mathbb{K}[[\mathbf{y}]]$ *is injective.*

We are interested in the decomposition of a function $h \in \mathscr{F}$ in terms of (generalized) eigenfunctions of the operators S_i. An eigenfunction of the operators S_i is a function $E \in \mathscr{F}$ such that $S_j(E) = \xi_j E$ for $j = 1, \ldots, n$ with $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{K}^n$. Generalized eigenfunctions of the operators S_i are functions $E_1, \ldots, E_\mu \in \mathscr{F}$ such that $S_j(E_k) = \xi_j E_k + \sum_{k'<k} m_{j,k'} E_{k'}$ for $k = 1, \ldots, \mu$ and $\xi_1, \ldots, \xi_n \in \mathbb{K}$.

The following proposition shows that if a function is a linear combination of generalized eigenfunctions, then its generating series is a sum of polynomial-exponential series.

Theorem 5.1.3 *Let $S_1, \ldots, S_n$ be commuting operators of $\mathscr{F}$. Let $E_{1,1}, \ldots, E_{1,\mu_1}, \ldots, E_{r,1}, \ldots, E_{r,\mu_r} \in \mathscr{F}$ be generalized eigenfunctions of $S_1, \ldots, S_n$ such that for $i = 1, \ldots, r$, $j = 1, \ldots, n$, $k = 1, \ldots, \mu_i$,*

$$S_j(E_{i,k}) = \xi_{i,j} E_{i,k} + \sum_{k'<k} m^{i,j}_{k',k} E_{i,k'}$$

with $\xi_i = (\xi_{i,1}, \ldots, \xi_{i,n}) \in \mathbb{K}^n$ pairwise distinct. If $h = \sum_{i=1}^r \sum_{k=1}^{\mu_i} h_{i,k} E_{i,k}$, then the generating series σ_h has a unique decomposition as:

$$\sigma_h(\mathbf{y}) \;=\; \sum_{i=1}^{r} \omega_i(\mathbf{y})\, \mathfrak{e}_{\xi_i}(\mathbf{y})$$

where $\omega_i(\mathbf{y}) \in \mathbb{K}[\mathbf{y}]$. If the regularity condition is satisfied, the decomposition uniquely determines the coefficients $h_{i,k}$ of the decomposition of $h \in \mathscr{F}$.

Proof. By Lemma 2.2.4, in a decomposition of series as a polynomial-exponential function $\sum_{i=1}^r \omega_i(\mathbf{y}) \mathfrak{e}_{\xi_i}(\mathbf{y})$, the polynomials $\omega_i(\mathbf{y}) \in \mathbb{K}[\mathbf{y}]$ and the support $\{\xi_1, \ldots, \xi_r\}$ are unique. Let $N_{i,j} = S_j - \xi_{i,j}\,\mathrm{Id}$ be the linear operator of $\mathfrak{e}_i = \langle E_{i,1}, \ldots, E_{i,\mu_i} \rangle$ such that $N_{i,j}(E_{j,k}) = \sum_{k'<k} m^i_{j,k'} E_{j,k'}$. By construction, $N_{i,j}$ is nilpotent of order $\leqslant \mu_i + 1$ and its matrix in the basis $\{E_{i,1}, \ldots, E_{i,\mu_i}\}$ of $\mathfrak{e}_i$ is $(m^{i,j}_{k,k'})_{k,k'}$ (with $m^{i,j}_{k,k'} = 0$ if $k \geqslant k'$). As the operators S_j restricted to $\mathfrak{e}_i$ are $\xi_{i,j}\,\mathrm{Id} + N_{i,j}$ and commute, we deduce that the operators $N_{i,j}$ commute for $j = 1, \ldots, n$. By the binomial expansion of $S^\alpha = S_1^{\alpha_1} \cdots S_n^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and the commutation of the matrices $N_{i,j}$, we have

$$S^\alpha(E_{i,k}) = \sum_{\beta \ll \alpha, \beta_j \leqslant \mu_j} \binom{\alpha}{\beta} \xi_i^{\alpha-\beta} N_i^\beta(E_{i,k}),$$

where $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}$ and $N_i^\beta = N_{i,1}^{\beta_1} \cdots N_{i,n}^{\beta_n}$. As $N_{i,j}$ is nilpotent of order $\mu_i + 1$, this sum involves at most $(\mu_i + 1)^n$ terms such that $\beta_j \leqslant \mu_j$, $j = 1, \ldots, n$.

The generating series of $E_{i,k}$ is then

$$\begin{aligned}
\sigma_{E_{i,k}}(\boldsymbol{y}) &= \sum_{\alpha\in\mathbb{N}^n}\sum_{\beta\ll\alpha,\beta_j\leqslant\mu_j}\Delta[N_i^{\beta}(E_{i,k})]\binom{\alpha}{\beta}\xi_i^{\alpha-\beta}\frac{\boldsymbol{y}^{\alpha}}{\alpha!}\\
&= \sum_{\beta_i\leqslant\mu_i}\Delta[N_i^{\beta}(E_{i,k})]\frac{\boldsymbol{y}^{\beta}}{\beta!}\sum_{\alpha'\in\mathbb{N}^n}\xi_i^{\alpha'}\frac{\boldsymbol{y}^{\alpha'}}{\alpha'!}\\
&= \sum_{\beta_i\leqslant\mu_i}\Delta[N_i^{\beta}(E_{i,k})]\frac{\boldsymbol{y}^{\beta}}{\beta!}\mathfrak{e}_{\xi_i}(\boldsymbol{y}) = \omega_{i,k}(\boldsymbol{y})\mathfrak{e}_{\xi_i}(\boldsymbol{y}),
\end{aligned}$$

using the relation $\frac{1}{\alpha!}\binom{\alpha}{\beta} = \frac{1}{\beta!}\frac{1}{(\alpha-\beta)!}$, exchanging the summation order and setting $\alpha' = \alpha-\beta$. We deduce that if $h = \sum_{i=1}^{r}\sum_{k=1}^{\mu_i} h_{i,k}E_{i,k}$, then $\sigma_h(\boldsymbol{y}) = \sum_{i=1}^{r}\omega_i(\boldsymbol{y})\mathfrak{e}_{\xi_i}(\boldsymbol{y})$ with $\omega_i(\boldsymbol{y}) = \sum_k h_{i,k}\omega_{i,k}(\boldsymbol{y}) \in \mathbb{K}[\boldsymbol{y}]$. If the regularity condition is satisfied, the map $h\in\mathscr{F}\mapsto \sigma_h(\boldsymbol{y})\in\mathbb{K}[[\boldsymbol{y}]]$ is injective and the polynomials $\omega_{i,k}(\boldsymbol{y})$ $k=1,\ldots,\mu_i$ are linearly independent. Therefore, the coefficients $h_{i,k}$, $k=1,\ldots,\mu_i$ are uniquely determined by the polynomial $\omega_i(\boldsymbol{y}) = \sum_k h_{i,k}\omega_{i,k}(\boldsymbol{y})$. □

Definition 5.1.4 *We say that the* completness *condition is satisfied if for any polynomial-exponential series $\omega(\boldsymbol{y})\mathfrak{e}_{\xi}(\boldsymbol{y})$ with $\omega(\boldsymbol{y})\in\mathbb{K}[\boldsymbol{y}]$ and $\xi\in\mathbb{K}^n$, there exists a linear combination $h\in\mathscr{F}$ of generalized eigenfunctions of the operators S_i, such that its generating function is $\omega(\boldsymbol{y})\mathfrak{e}_{\xi}(\boldsymbol{y})$.*

Under the completness condition and the regularity condition, any function $h\in\mathscr{F}$ with a generating series of finite rank can be decomposed into a linear combination of eigenfunctions. We analyse several cases, for which this framework applies.

5.2 Convolution of finite rank

Let $\mathscr{E} = C^{\infty}(\mathbb{R}^n)$, $\mathscr{S}$ be the set of functions in $\mathscr{E}$ with fast decrease at infinity ($\forall f\in\mathscr{S}, \forall p\in\mathbb{C}[\boldsymbol{x}]$, $|pf|$ is bounded on $\mathbb{R}^n$), $\mathscr{O}_M$ be the set of functions in $\mathscr{E}$ with slow increase at infinity ($\forall f\in\mathscr{O}_M, |f(\boldsymbol{x})| < C(1+|\boldsymbol{x}|)^N$ for some $C\in\mathbb{R}, N\in\mathbb{N}$), $\mathscr{E}'$ be the set of distributions with compact support (dual to $\mathscr{E}$), $\mathscr{S}'$ be the set of tempered distribution (dual to $\mathscr{S}$) and $\mathscr{O}'_C$ be the space of distributions with rapid decrease at infinity (see [Sch66]).

In this problem, we consider the following space and operators:

- $\mathscr{F} = \mathscr{O}'_C$ is the space of distributions with rapid decrease at infinity;
- $S_i : h(\boldsymbol{x})\in\mathscr{O}'_C \mapsto x_i h(\boldsymbol{x})\in\mathscr{O}'_C$ is the multiplication by x_j;
- $\Delta : h(\boldsymbol{x})\in\mathscr{O}'_C \mapsto \int h(\boldsymbol{x})d\boldsymbol{x}\in\mathbb{C}$.

For any $h \in \mathscr{O}'_C$, for any $\alpha \in \mathbb{N}^n$,

$$\Delta[S^\alpha(h)] = \int \boldsymbol{x}^\alpha h(\boldsymbol{x}) d\boldsymbol{x}$$

is the α^{th} moment of h. For $h \in \mathscr{O}'_C$ and $\sigma_h = \sum_{\alpha\in\mathbb{N}^n} \int h(\boldsymbol{x})\boldsymbol{x}^\alpha \frac{\boldsymbol{y}^\alpha}{\alpha!} d\boldsymbol{x}$ its generating series, we verify that $\forall p \in \mathbb{C}[\boldsymbol{x}]$, $\langle \sigma_h \mid p\rangle = \int h(\boldsymbol{x})p(\boldsymbol{x})d\boldsymbol{x}$ (i.e. the distribution h applied to p). We check that

- the operators S_j are well defined and commute
- a Dirac measure δ_ξ with $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ is an eigenfunction of S_j: $S_j(\delta_\xi) = \xi_j \delta_\xi$. Similarly for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and
- the Dirac derivation $\delta_\xi^{(\alpha)}$ ($\forall f \in C^\infty(\Omega)$, $\langle \delta_\xi^{(\alpha)}, f\rangle = (-1)^{|\alpha|}\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}(f)(\xi)$) satisfies
$$S_i(\delta_\xi^{(\alpha)}) = x_i \delta_\xi^{(\alpha)} = \xi_i \delta_\xi^{(\alpha)} + \delta_\xi^{(\alpha - e_i)}$$
with the convention that $\delta_\xi^{(\alpha-e_i)} = 0$ if $\alpha_i = 0$. It is a generalized eigenfunction of the operators S_j.

By the relation (5.2), the generating series of $\delta_\xi^{(\alpha)}$ is

$$\sigma_{\delta_\xi^{(\alpha)}} = \langle \delta_\xi^{(\alpha)}, \boldsymbol{e}^{\boldsymbol{x}\cdot\boldsymbol{y}}\rangle = \boldsymbol{y}^\alpha \boldsymbol{e}_\xi(\boldsymbol{y}).$$

This shows that the completeness condition is satisfied.

To check the regularity condition, we use the *Fourier transform* $\mathscr{F} : f \in \mathscr{O}_M \mapsto \int f(\boldsymbol{x}) e^{-i\boldsymbol{x}\cdot\boldsymbol{z}} d\boldsymbol{x} \in \mathscr{O}'_C$. It is a bijection between $\mathscr{O}_M$ and $\mathscr{O}'_C$ (see [Sch66][Théorème XV]). Its inverse is $\mathscr{F}^{-1} : f \in \mathscr{O}'_C \mapsto (2\pi)^n \int f(\boldsymbol{x}) e^{i\boldsymbol{x}\cdot\boldsymbol{z}} d\boldsymbol{x} \in \mathscr{O}_M$. Let $\iota : f(\boldsymbol{y}) \in \mathbb{C}[[\boldsymbol{y}]] \mapsto f(\boldsymbol{i}\boldsymbol{y}) \in \mathbb{C}[[\boldsymbol{y}]]$.

The generating series of $f \in \mathscr{O}'_C$ is

$$\sigma_f(\boldsymbol{i}\boldsymbol{y}) = \iota \circ \sigma_f(\boldsymbol{y}) = \sum_{\alpha\in\mathbb{N}^n} \int f(\boldsymbol{x})\boldsymbol{x}^\alpha \frac{\boldsymbol{i}^{|\alpha|}\boldsymbol{y}^\alpha}{\alpha!} d\boldsymbol{x} = \int f(\boldsymbol{x}) \boldsymbol{e}^{i\boldsymbol{x}\cdot\boldsymbol{y}} d\boldsymbol{x} = (2\pi)^n \mathscr{F}^{-1}(f). \quad (5.2)$$

This shows that the map $f \in \mathscr{O}'_C \mapsto \sigma_f \in \mathbb{C}[[\boldsymbol{y}]]$ is injective and the regularity condition is satisfied.

For $f \in \mathscr{O}'_C$, the Hankel operator H_{σ_f} is such that $\forall g \in \mathbb{C}[\boldsymbol{x}]$,

$$\begin{aligned} H_{\iota\circ\sigma_f}(g) &= \sum_{\alpha\in\mathbb{N}^n} \int f(\boldsymbol{x}) g(\boldsymbol{x}) \boldsymbol{x}^\alpha \frac{\boldsymbol{i}^{|\alpha|}\boldsymbol{y}^\alpha}{\alpha!} d\boldsymbol{x} \\ &= \int f(\boldsymbol{x}) g(\boldsymbol{x}) \boldsymbol{e}^{i\boldsymbol{x}\cdot\boldsymbol{y}} d\boldsymbol{x} = (2\pi)^n \mathscr{F}^{-1}(fg). \end{aligned}$$

Using Relation (5.2), we rewrite it as $\forall g \in \mathbb{C}[\boldsymbol{x}]$,

$$H_{\mathscr{F}^{-1}(f)}(g) = \mathscr{F}^{-1}(f\,g) \tag{5.3}$$

with $\varphi = \mathscr{F}^{-1}(f) \in \mathscr{O}_M$.

From this relation, we see that the operator $H_{\mathscr{F}^{-1}(f)}$ can be extended by continuity to an operator $H_{\mathscr{F}^{-1}(f)} : \mathscr{O}_M \mapsto \mathscr{O}_M$.

The Hankel operator $H_{\iota\circ\sigma}$ (or $H_{\mathscr{F}^{-1}(f)}$) can be related to integral operators on functions defined in terms of convolution products or cross-correlation. For $\varphi \in \mathscr{S}'$, the convolution with a distribution $\psi \in \mathscr{O}'_C$ is well-defined [Sch66]. The convolution operator associated to φ on $\mathscr{O}'_C$ is:

$$\mathfrak{H}_\varphi : \psi \in \mathscr{O}'_C \mapsto \varphi \star \psi = \int \varphi(\boldsymbol{x}-\boldsymbol{t})\psi(\boldsymbol{t})d\boldsymbol{t} \in \mathscr{S}'.$$

The image of an element $\psi \in \mathscr{O}'_C$ is a tempered distribution in $\mathscr{S}'$. The distribution φ is the *symbol* of the operator $\mathfrak{H}_\varphi$.

Using the property that $\forall \varphi \in \mathscr{S}', \forall \psi \in \mathscr{O}'_C, \mathscr{F}(\varphi \star \psi) = \mathscr{F}(\varphi)\mathscr{F}(\psi) \in \mathscr{S}'$ and the relation (5.3), we have for any $\psi \in \mathscr{O}'_C$,

$$H_{\mathscr{F}^{-1}(f)}(g) = \mathscr{F}^{-1}(f\,g) = \varphi \star \psi = \mathfrak{H}_\varphi(\psi),$$

with $f = \mathscr{F}(\varphi) \in \mathscr{S}'$, $g = \mathscr{F}(\psi) \in \mathscr{O}_M$. We deduce that

$$\mathfrak{H}_\varphi = H_\varphi \circ \mathscr{F} \tag{5.4}$$

with $H_\varphi : g \in \mathscr{O}_M \mapsto \mathscr{F}^{-1}(\mathscr{F}(\varphi)g) \in \mathscr{S}'$.

In the case where $\varphi \in \mathscr{P}ol\mathscr{E}xp \cap \mathscr{O}_M$, the operator is of finite rank:

Proposition 5.2.1 *Let $\varphi = \omega(\boldsymbol{y})\boldsymbol{e}_{i\xi}(\boldsymbol{y})$ with $\omega \in \mathbb{C}[\boldsymbol{y}]$ and $\xi \in \mathbb{R}^n$. Then* $\operatorname{rank} \mathfrak{H}_\varphi \leqslant \mu(\omega)$.

Proof. By Taylor expansion of the polynomial ω at $\boldsymbol{x}$, we have $\forall \psi \in \mathscr{O}'_C$

$$\begin{aligned} \mathfrak{H}_\varphi(\psi) &= \int \omega(\boldsymbol{x}-\boldsymbol{t})e^{i\xi\cdot(\boldsymbol{x}-\boldsymbol{t})}\psi(\boldsymbol{t})d\boldsymbol{t} \\ &= \sum_{\alpha\in\mathbb{N}^n} \partial^\alpha(\omega)(\boldsymbol{x})e^{i\xi\cdot\boldsymbol{x}} \int (-1)^\alpha \frac{\boldsymbol{t}^\alpha}{\alpha!}\psi(\boldsymbol{t})e^{-i\xi\cdot\boldsymbol{t}}d\boldsymbol{t}. \end{aligned}$$

This shows that $\mathfrak{H}_\varphi(\psi)$ belongs to the space spanned by $\partial^\alpha(\omega)(\boldsymbol{x})e^{\xi\cdot\boldsymbol{x}}$ for $\alpha \in \mathbb{N}^n$, which is of dimension $\mu(\omega)$ and thus $\operatorname{rank} \mathfrak{H}_\varphi \leqslant \mu(\omega)$. □

The converse is also true:

Theorem 5.2.2 *Suppose that $\varphi \in \mathscr{S}'$ is such that the convolution operator $\mathfrak{H}_\varphi$ is of finite rank r. Then its symbol φ is of the form*

$$\varphi = \sum_{i=1}^{r'} \omega_i(\boldsymbol{y}) \boldsymbol{e}_{i\xi_i}(\boldsymbol{y}).$$

with $\xi_i = (\xi_{i,1}, \ldots, \xi_{i,n}) \in \mathbb{R}^n$, $\omega_i(\boldsymbol{y}) \in \mathbb{C}[\boldsymbol{y}]$. The rank r of $\mathfrak{H}_\varphi$ is the sum of the dimension of the vector spaces spanned by $\omega_i(\boldsymbol{y})$ and all its derivatives $\partial_{\boldsymbol{y}}^{\gamma} \omega_i(\boldsymbol{y})$, $\gamma \in \mathbb{N}^n$.

Proof. Since $\mathscr{F}$ is a bijection between $\mathscr{O}'_C$ and $\mathscr{O}_M$, the relation (5.4) implies that $\mathfrak{H}_\varphi$ is of finite rank r, if and only if, $H_\varphi : \mathscr{O}_M \mapsto \mathscr{O}_M$ is of rank r. As the restriction of H_φ to the set of polynomials $\mathbb{C}[\boldsymbol{x}] \subset \mathscr{O}_M$ is of rank $\tilde{r} \leq r = \operatorname{rank} \mathfrak{H}_\varphi$, Theorem 4.2.2 implies that

$$\varphi = \sum_{i=1}^{r'} \sum_{\alpha \in A_i} \omega_{i,\alpha} \boldsymbol{y}^{\alpha} \boldsymbol{e}_{\xi'_i}(\boldsymbol{y})$$

with $\xi'_i \in \mathbb{C}^n$ distincts, $A_i \subset \mathbb{N}^n$ finite and $\tilde{r} = \sum_{i=1}^{r'} \mu(\omega_i)$ where $\mu(\omega_i)$ is the dimension of the inverse system of $\omega_i = \sum_{\alpha \in A_i} \omega_{i,\alpha} \boldsymbol{y}^{\alpha}$, spanned by $\omega_i(\boldsymbol{y})$ and all its derivatives. As $\varphi \in \mathscr{S}'$ is a distribution with slow increase at infinity, we have $\xi'_i = \boldsymbol{i}\xi_i$ with $\xi_i \in \mathbb{R}^n$.

By Proposition 5.2.1, we have $r = \operatorname{rank} \mathfrak{H}_\varphi \leqslant \sum_{i=1}^{r'} \mu(\omega_i) = \tilde{r}$. This shows that $\operatorname{rank} \mathfrak{H}_\varphi = \sum_{i=1}^{r'} \mu(\omega_i)$ and concludes the proof of the theorem. □

We can derive a similar result for the convolution by functions or distributions with support in a bounded domain Ω of $\mathbb{R}^n$. The main ingredient is the decomposition $\mathfrak{H}_\varphi = H_\varphi \circ \mathscr{F}$, which extends the construction used in [Roc87] for Hankel and Toeplitz operators on $L^2(I)$ where I is a bounded interval in $\mathbb{R}$.

By the generalized Paley-Wiener theorem (see [Sch66][Théorème XVI]), the Fourier transform $\mathscr{F}$ is a bijection between the set $\mathscr{E}'$ of distributions with a compact support and the set of continuous functions $f \in C(\mathbb{R}^n)$ with an analytic extension of exponential type (there exists $A \in \mathbb{R}, C \in \mathbb{R}_+$ such that $\forall \boldsymbol{z} \in \mathbb{C}^n, |f(\boldsymbol{z})| \leqslant C e^{A(|z_1|+\cdots+|z_n|)}$). Let us denote by $\mathscr{E}'(\Omega)$ the set of distributions with a support in Ω, and by $\mathscr{P}\mathscr{W}(\Omega) = \{\mathscr{F}(\varphi) \mid \varphi \in \mathscr{E}'(\Omega)\}$ the set of their Fourier transforms.

Theorem 5.2.3 *Let Ω, Ξ be open bounded domains of $\mathbb{R}^n$ with , $\Upsilon = \Xi + \Omega \subset \mathbb{R}^n$ and $\varphi \in \mathscr{E}'(\Omega)$. The convolution operator*

$$\mathfrak{H}_\varphi : \psi \in \mathscr{E}'(\Xi) \mapsto \int \varphi(x-t)\psi(t)\,\mathrm{d}t \in \mathscr{E}'(\Upsilon)$$

is of finite rank r, if and only if, the symbol φ is of the form

$$\varphi = \mathbb{1}_\Omega \sum_{i=1}^{r'} \omega_i(\boldsymbol{y}) \boldsymbol{e}_{\xi_i}(\boldsymbol{y})$$

where $\xi_i = (\xi_{i,1}, \ldots, \xi_{i,n}) \in \mathbb{C}^n$, $\omega_i(\mathbf{y}) \in \mathbb{C}[\mathbf{y}]$. The rank r of $\mathfrak{H}_\varphi$ is the sum of the dimensions $\mu(\omega_i)$ of the vector spaces spanned by $\omega_i(\mathbf{y})$ and all the derivatives $\partial_{\mathbf{y}}^{\gamma} \omega_i(\mathbf{y})$, $\gamma \in \mathbb{N}^n$.

Proof. Using the relations $\forall \varphi \in \mathscr{E}'(\Omega), \psi \in \mathscr{E}'(\Xi)$,

$$\mathscr{F}(\varphi \star \psi) = \mathscr{F}(\varphi)\mathscr{F}(\psi),$$

and (5.3), we still have the decomposition

$$\mathfrak{H}_\varphi = H_\varphi \circ \mathscr{F}.$$

with $H_\varphi : g \in \mathscr{PW}(\Xi) \to \mathscr{F}^{-1}(\mathscr{F}(\varphi)g) \in \mathscr{E}'(\Upsilon)$. Thus $\mathfrak{H}_\varphi$ is of finite rank r, if and only if, H_φ is of finite rank r. As the rank of the restriction of H_φ to $\mathbb{C}[\mathbf{x}] \subset \mathscr{PW}(\Xi)$ is at most r, we conclude by using Theorem 4.2.2, a result similar to Proposition 5.2.1 for elements $\psi \in \mathscr{E}'(\Xi)$ and the relation $\mathscr{F}^{-1}(\mathscr{F}(\varphi)) = \varphi$ on Ω. $\square$

Similar results also apply for the cross-correlation operator defined as

$$\tilde{\mathfrak{H}}_\varphi : \psi \in \mathscr{E}' \mapsto \varphi * \psi = \int \varphi(x+t)\bar{\psi}(t)\,\mathrm{d}t \in \mathscr{S}'.$$

Using the relation $\mathscr{F}(\varphi * \psi) = \mathscr{F}(\varphi)\bar{\mathscr{F}}(\psi)$ (with $\bar{\mathscr{F}} = \varsigma \circ \mathscr{F}$ where $\varsigma : z \in \mathbb{C} \mapsto \bar{z} \in \mathbb{C}$ is the complex conjugation), we have $\tilde{\mathfrak{H}}_\varphi = H_\varphi \circ \bar{\mathscr{F}}$. As $\bar{\mathscr{F}}^{-1} = \mathscr{F}^{-1} \circ \varsigma$, we deduce that $\tilde{\mathfrak{H}}_\varphi$ and H_φ have the same rank and the same type of decomposition of the symbol φ holds when $\tilde{\mathfrak{H}}_\varphi$ is of finite rank.

Remark 5.2.4 *To compute the decomposition of $\varphi \in \mathscr{S}'$ (resp. $\varphi \in \mathscr{E}'(\Omega)$) as a polynomial exponential function, we first compute the Taylor coefficients of $\sigma_{\mathscr{F}(\varphi)} = \mathfrak{H}_\varphi(1)$, that is, the values $\sigma_\alpha = (-\mathbf{i})^{|\alpha|}\mathscr{F}(\mathbf{x}^\alpha \varphi)(\mathbf{0})$ for some $\alpha \in \mathbf{a} \subset \mathbb{N}^n$ and apply the decomposition algorithm 4.6.2 to the (truncated) sequence $(\sigma_\alpha)_{\alpha \in \mathbf{a}}$.*

5.3 Dirac measures from Fourier coefficients

We consider here the problem of reconstruction of functions or distributions from Fourier coefficients. Let $T = (T_1, \ldots, T_n) \in \mathbb{R}_+^n$ and $\Omega = \prod_{i=1}^n \left[-\frac{2\pi T_i}{2}, \frac{2\pi T_i}{2}\right] \subset \mathbb{R}^n$. We take:

- $\mathscr{F} = L^2(\Omega)$;
- $S_i : h(\mathbf{x}) \in L^2(\Omega) \mapsto e^{2\pi \frac{x_i}{T_i}} h(\mathbf{x}) \in L^2(\Omega)$ is the multiplication by $e^{2\pi \frac{x_i}{T_i}}$;
- $\Delta : h(\mathbf{x}) \in \mathscr{O}'_C \mapsto \int h(\mathbf{x}) d\mathbf{x} \in \mathbb{C}$.

For $f \in \mathscr{E}'(\Omega)$ with a support in Ω and $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n$, the γ-th Fourier coefficient of f is

$$\sigma_\gamma = \frac{1}{\prod_{j=1}^n T_j} \mathscr{F}(f)\left(2\pi \frac{\gamma_1}{T_1}, \ldots, 2\pi \frac{\gamma_n}{T_n}\right) = \frac{1}{\prod_{j=1}^n T_j} \int f(\boldsymbol{x}) e^{-2\pi \boldsymbol{i} \sum_{j=1}^n \frac{\gamma_j x_j}{T_j}} d\boldsymbol{x}$$

Let $\sigma = (\sigma_\gamma)_{\gamma \in \mathbb{Z}^n}$ be the sequence of the Fourier coefficients. The discrete convolution operator associated to σ is $\Phi_\sigma : (\rho_\beta)_{\beta \in \mathbb{Z}^n} \in L^2(\mathbb{Z}^n) \mapsto \left(\sum_\beta \sigma_{\alpha-\beta} \rho_\beta\right)_{\alpha \in \mathbb{Z}^n} \in L^2(\mathbb{Z}^n)$. The discrete cross-correlation operator of σ is $\Gamma_\sigma : (\rho_\beta)_{\beta \in \mathbb{Z}^n} \in L^2(\mathbb{Z}^n) \mapsto \left(\sum_\beta \sigma_{\alpha+\beta} \rho_\beta\right)_{\alpha \in \mathbb{Z}^n} \in L^2(\mathbb{Z}^n)$. It is obtained from Γ_σ by composition by $\mathscr{R} : (r_\beta)_{\beta \in \mathbb{Z}^n} \in L^2(\mathbb{Z}^n) \mapsto (\rho_{-\beta})_{\beta \in \mathbb{Z}^n} \in L^2(\mathbb{Z}^n)$: $\Gamma_\sigma = \Phi_\sigma \circ \mathscr{R}$.

A decomposition similar to the previous section also holds:

Theorem 5.3.1 *Let $f \in L^2(\Omega)$ and let $\sigma = (\sigma_\gamma)_{\gamma \in \mathbb{Z}^n}$ be its sequence of Fourier coefficients. The discrete convolution (resp. cross-correlation) operator Φ_σ (resp. Γ_σ) is of finite rank if and only if*

$$f = \sum_{i=1}^{r'} \sum_{\alpha \in A_i \subset \mathbb{N}^n} \omega_{i,\alpha} \delta_{\xi_i}^{(\alpha)}$$

where

- *$\xi_i = (\xi_{i,1}, \ldots, \xi_{i,n}) \in \Omega$, $\omega_{i,\alpha} \in \mathbb{C}$, $A_i \subset \mathbb{N}^n$ is finite,*
- *the rank of Γ_σ is the sum of the dimensions $\mu(\omega_i)$ of the vector spaces spanned by $\omega_i(\boldsymbol{y}) = \sum_{\alpha \in A_i} \omega_{i,\alpha} \boldsymbol{y}^\alpha$ and all the derivatives $\partial_{\boldsymbol{y}}^\gamma(\omega_i)$, $\gamma \in \mathbb{N}^n$.*

Proof. Let $S : f \in L^2(\Omega) \mapsto (\sigma_\gamma)_{\gamma \in \mathbb{Z}^n} \in L^2(\mathbb{Z}^n)$ be the discrete Fourier transform where $\sigma_\gamma = \frac{1}{\prod_{j=1}^n T_j} \mathscr{F}(f)\left(2\pi \frac{\gamma_1}{T_1}, \ldots, 2\pi \frac{\gamma_n}{T_n}\right)$. Its inverse is $S^{-1} : \sigma = (\sigma_\gamma)_{\gamma \in \mathbb{Z}^n} \in L^2(\mathbb{Z}^n) \mapsto \sum_{\alpha \in \mathbb{Z}^n} \sigma_\gamma \mathbb{1}_\Omega e^{2\pi \boldsymbol{i} \sum_{j=1}^n \frac{\gamma_j x_j}{T_j}} \in L^2(\Omega)$. As the discrete Fourier transform exchanges the convolution and the product, using Relation (5.3), we have $\forall \sigma, \rho \in L^2(\mathbb{Z}^n)$,

$$\Phi_\sigma(\rho) = S(S^{-1}(\sigma) S^{-1}(\rho)) = S(f g) = S \circ \mathscr{F} \circ \mathscr{F}^{-1}(f g) = S \circ \mathscr{F} \circ H_{\mathscr{F}^{-1}(f)}(g)$$

where $f = S^{-1}(\sigma), g = S^{-1}(\rho) \in L^2(\Omega)$ and $H_{\mathscr{F}^{-1}(f)} : g \in L^2(\Omega) \mapsto \mathscr{F}^{-1}(f g) \in \mathscr{PW}(\Omega)$. We deduce that

$$\Phi_\sigma = S \circ \mathscr{F} \circ H_{\mathscr{F}^{-1} \circ S^{-1}(\sigma)} \circ S^{-1}.$$

As S is an isometry between $L^2(\mathbb{Z}^n)$ and $L^2(\Omega)$ and $\mathscr{F}$ is an isomorphism between $L^2(\Omega)$ and $\mathscr{PW}(\Omega)$, $\Phi_\sigma = S \circ \mathscr{F} \circ H_{\mathscr{F}^{-1} \circ S^{-1}(\sigma)} \circ S^{-1}$ and $H_{\mathscr{F}^{-1} \circ S^{-1}(\sigma)}$ have the same rank.

As $\mathbb{C}[\boldsymbol{x}] \subset \mathrm{PW}(\Omega)$, we deduce from Theorem 4.2.2 that

$$\mathscr{F}^{-1} \circ S^{-1}(\sigma) = \sum_{i=1}^{r'} \tilde{\omega}_i(\boldsymbol{y}) \boldsymbol{e}_{\xi_i}(\boldsymbol{y})$$

where $\tilde{\xi}_i = (\tilde{\xi}_{i,1}, \ldots, \tilde{\xi}_{i,n}) \in \mathbb{C}^n$, $\tilde{\omega}_i(\boldsymbol{y}) = \sum_{\alpha \in A_i} \tilde{\omega}_{i,\alpha} \boldsymbol{y}^\alpha \in \mathbb{C}[\boldsymbol{z}]$. Using a result similar to Proposition 5.2.1 for the elements $\psi \in L^2(\Omega)$, we deduce that the rank r of Φ_σ is $r = \sum_{i=1}^{r'} \mu(\tilde{\omega}_i)$. Consequently,

$$f = S^{-1}(\sigma) = \mathscr{F}\left(\sum_{i=1}^{r'} \sum_{\alpha \in A_i} \tilde{\omega}_{i,\alpha} \boldsymbol{y}^\alpha \boldsymbol{e}_{\tilde{\xi}_i}(\boldsymbol{y})\right) = (2\pi)^n \sum_{i=1}^{r'} \sum_{\alpha \in A_i \subset \mathbb{N}^n} \boldsymbol{i}^{|\alpha|} \tilde{\omega}_{i,\alpha} \delta^{(\alpha)}_{\boldsymbol{i}\tilde{\xi}_i}.$$

As the support of f is in Ω, we have $\xi_i = \boldsymbol{i}\tilde{\xi}_i \in \Omega$. We deduce the decomposition of f with $\omega_i = (2\pi)^n \sum_{i=1}^{r'} \sum_{\alpha \in A_i \subset \mathbb{N}^n} \boldsymbol{i}^{|\alpha|} \tilde{\omega}_{i,\alpha} \boldsymbol{y}^\alpha$.

The dimension $\mu(\tilde{\omega}_i)$ of the vector space spanned by $\tilde{\omega}_i(\boldsymbol{y}) = \sum_{\alpha \in A_i} \omega_{i,\alpha} \boldsymbol{y}^\alpha$ and all its derivatives is the same as the dimension $\mu(\omega_i)$ of the space spanned by $\omega_i(\boldsymbol{y}) = (2\pi)^n \sum_{\alpha \in A_i} \omega_{i,\alpha} \boldsymbol{i}^{|\alpha|} \boldsymbol{y}^\alpha$ and all its derivatives, since $\omega_i(\boldsymbol{y}) = (2\pi)^n \tilde{\omega}_i(\boldsymbol{i}\boldsymbol{y})$. Therefore, $\operatorname{rank} \Phi_\sigma = r = \sum_{i=1}^{r'} \mu(\tilde{\omega}_i) = \sum_{i=1}^{r'} \mu(\omega_i)$. This concludes the proof of the theorem. □

Remark 5.3.2 *To compute the decomposition of $f \in L^2(\Omega)$ as a weighted sum of Dirac measures and derivates, we apply the decomposition algorithm 4.6.2 to the (truncated) sequence of Fourier coefficients $(\sigma_\alpha)_{\alpha \in \boldsymbol{a}}$ for some subset $\boldsymbol{a} \subset \mathbb{N}^n$. The polynomial-exponential decomposition $\varphi = \sum_{i=1}^{r'} \sum_{\alpha \in A_i} \tilde{\omega}_{i,\alpha} \boldsymbol{y}^\alpha \boldsymbol{e}_{\tilde{\xi}_i}(\boldsymbol{y})$, from which we deduce the decomposition $f = (2\pi)^n \sum_{i=1}^{r'} \sum_{\alpha \in A_i \subset \mathbb{N}^n} \boldsymbol{i}^{|\alpha|} \tilde{\omega}_{i,\alpha} \delta^{(\alpha)}_{\boldsymbol{i}\tilde{\xi}_i}$.*

5.4 Polynomial-exponential sums from values

In this problem, we are interested in reconstructing a function in $C^\infty(\mathbb{R}^n)$ from sampled values. We take

- $\mathscr{F} = C^\infty(\mathbb{R}^n)$,
- $S_j : h(x) \mapsto h(x_1, \ldots, x_{j-1}, x_j + 1, x_{j+1}, \ldots, x_n)$ the shift operator of x_j by 1,
- $\Delta : h(x) \mapsto \Delta[h] = h(0)$ the evaluation at 0.

The generating series of h is

$$\sigma_h(\boldsymbol{y}) = \sum_{\alpha \in \mathbb{N}^n} h(\alpha_1, \ldots, \alpha_n) \frac{\boldsymbol{y}^\alpha}{\alpha!} = \sum_{\alpha \in \mathbb{N}^n} h(\alpha) \frac{\boldsymbol{y}^\alpha}{\alpha!}.$$

The operators S_j are commuting and we have $S_j(e^{\boldsymbol{f} \cdot \boldsymbol{x}}) = \xi_j e^{\boldsymbol{f} \cdot \boldsymbol{x}}$ where $\boldsymbol{f} = (f_1, \ldots, f_n) \in \mathbb{C}^n$ and $\xi_j = e^{f_j}$. The generating series associated to $e^{\boldsymbol{f} \cdot \boldsymbol{x}}$ is $\boldsymbol{e}_\xi(\boldsymbol{y})$ where $\xi = (\xi_1, \ldots, \xi_n) = (e^{f_1}, \ldots, e^{f_n})$.

Similarly for any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}$, $S_j(\boldsymbol{x}^\alpha e^{\boldsymbol{f}\cdot \boldsymbol{x}}) = \xi_j \sum_{i=0}^{\alpha_j} \binom{\alpha_j}{i} x_j^i \prod_{i\neq j} x_i^{\alpha_i} e^{\boldsymbol{f}\cdot\boldsymbol{x}}$, which shows that the function $\boldsymbol{x}^\alpha e^{f\cdot x}$ is a generalized eigenfunction of the operators S_j. Its generating series is

$$\sigma_{\boldsymbol{x}^\alpha e^{f\cdot x}}(\boldsymbol{y}) = \sum_{\beta\in\mathbb{N}^n} \beta^\alpha \xi^\beta \frac{\boldsymbol{y}^\beta}{\beta!}. \tag{5.5}$$

Let $b_\alpha(\boldsymbol{y}) = \binom{y_1}{\alpha_1}\cdots\binom{y_n}{\alpha_n}$ be the Macaulay binomial polynomial with $\binom{y_i}{\alpha_i} = \frac{1}{\alpha_i!} y_i(y_i - 1)\cdots(y_i - \alpha_i + 1)$, which roots are $0, \ldots, \alpha_i - 1$. It satisfies the following relations:

$$\sum_{\beta\in\mathbb{N}^n} b_\alpha(\beta)\xi^\beta \frac{\boldsymbol{y}^\beta}{\beta!} = \sum_{\beta\gg\alpha} b_\alpha(\beta)\xi^\beta \frac{\boldsymbol{y}^\beta}{\beta!} = \sum_{\beta\gg\alpha} \frac{1}{\alpha!}\xi^\beta \frac{\boldsymbol{y}^\beta}{(\beta-\alpha)!} = \frac{1}{\alpha!}\xi^\alpha \boldsymbol{y}^\alpha \boldsymbol{e}_\xi(\boldsymbol{y}).$$

As $\boldsymbol{y}^\alpha = \sum_{\alpha'\ll\alpha} m_{\alpha',\alpha} b_{\alpha'}(\boldsymbol{y})$ for some coefficients $m_{\alpha',\alpha} \in \mathbb{Q}$ such that $m_{\alpha,\alpha} = 1$, we have

$$\sigma_{\boldsymbol{x}^\alpha e^{f\cdot x}}(\boldsymbol{y}) = \left(\sum_{\alpha'\ll\alpha} m_{\alpha',\alpha} \xi^{\alpha'} \frac{\boldsymbol{y}^{\alpha'}}{\alpha'!}\right) \boldsymbol{e}_\xi(\boldsymbol{y}) = \omega_\alpha(\boldsymbol{y}) \boldsymbol{e}_\xi(\boldsymbol{y}). \tag{5.6}$$

The monomials of $\omega_\alpha(\boldsymbol{y})$ are among the monomials $\boldsymbol{y}^{\alpha'} = y_1^{\alpha'_1}\cdots y_n^{\alpha'_n}$ such that $0 \leqslant \alpha'_i \leqslant \alpha_i$, which divide $\boldsymbol{y}^\alpha$. As the coefficient of $\boldsymbol{y}^\alpha$ in $\omega_\alpha(\boldsymbol{y})$ is 1, we deduce that $(\omega_\alpha)_{\alpha\in\mathbb{N}^n}$ is a basis of $\mathbb{C}[\boldsymbol{y}]$ and the completeness property is satisfied.

Let $h = (h(\alpha))_{\alpha\in\mathbb{N}^n}$. The Hankel operator H_h is such that $\forall p = \sum_\beta p_\beta \boldsymbol{x}^\beta \in \mathbb{C}[\boldsymbol{x}]$,

$$H_h(p) = \sum_{\alpha\in\mathbb{N}^n} \left(\sum_\beta h(\alpha+\beta)p_\beta\right) \frac{\boldsymbol{y}^\alpha}{\alpha!}$$

Identifying the series $\sigma(\boldsymbol{y}) = \sum_{\alpha\in\mathbb{N}^n} \sigma_\alpha \frac{\boldsymbol{y}^\alpha}{\alpha!} \in \mathbb{C}[[\boldsymbol{y}]]$ with the multi-index sequence $(\sigma_\alpha)_{\alpha\in\mathbb{N}^n}$ and a polynomial $p = \sum_{\alpha\in A} p_\alpha \boldsymbol{x}^\alpha$ with the sequence $(p_\alpha)_{\alpha\in\mathbb{N}^n}$ $L_0(\mathbb{N}^n)$ of finite support, the operator H_h corresponds to the discrete cross-correlation operator by the sequence h. This operator can be extended to sequences h, p are in $L^2(\mathbb{N}^n)$.

Theorem 5.4.1 *Let $h \in C^\infty(\mathbb{R}^n)$. The discrete cross-correlation operator $\Gamma_h : p \in L^2(\mathbb{N}^n) \mapsto h \star p = \left(\sum_\beta h(\alpha+\beta)p_\beta\right)_{\alpha\in\mathbb{N}^n} \in L^2(\mathbb{N}^n)$ is of finite rank, if and only if,*

$$h(\boldsymbol{x}) = \sum_{i=1}^{r'} g_i(\boldsymbol{x}) e^{f_i\cdot \boldsymbol{x}} + r(\boldsymbol{x})$$

where

- $f_i = (f_{i,1}, \ldots, f_{i,n}) \in \mathbb{C}^n$, $g_i(\boldsymbol{x}) \in \mathbb{C}[\boldsymbol{x}]$,
- $r(\boldsymbol{x}) \in C^\infty(\mathbb{R}^n)$ *such that* $r(\alpha) = 0$, $\forall \alpha \in \mathbb{N}^n$,

- *The rank of* Γ_h *is the sum of the dimension* $\mu(g_i)$ *of the vector space spanned by* $g_i(\boldsymbol{x})$ *and all its derivatives* $\partial_{\boldsymbol{x}}^{\alpha} g_i$, $\alpha \in \mathbb{N}^n$.

Proof. Since H_h is of finite rank, Theorem 4.2.2 implies that

$$\sigma_h = \sum_{\alpha \in \mathbb{N}^n} h(\alpha) \frac{\boldsymbol{y}^\alpha}{\alpha!} = \sum_{i=1}^{r'} \omega_i(\boldsymbol{y}) \boldsymbol{e}_{\xi_i}(\boldsymbol{y})$$

where $\xi_i \in \mathbb{C}^n$, $\omega_i(\boldsymbol{x}) \in \mathbb{C}[\boldsymbol{x}]$ and rank $H_{\sigma_h} = \sum_{i=1}^{r'} \mu(\omega_i)$. Let $\boldsymbol{f}_i = (f_{i,1}, \dots, f_{i,n}) \in \mathbb{C}^n$ such that $\xi_i = (e^{f_{i,1}}, \dots, e^{f_{i,n}})$ and $g_{i,\alpha} \in \mathbb{C}$ for $\alpha \in A_i \subset \mathbb{N}^n$ such that

$$\omega_i(\boldsymbol{y}) = \sum_{\alpha \in A_i} g_{i,\alpha} \omega_\alpha(\boldsymbol{y}).$$

By the relation (5.6), the generating series of $r(\boldsymbol{x}) = h - \sum_{i=1}^{r} \sum_{\alpha \in A_i} g_{i,\alpha} \boldsymbol{x}^\alpha e^{\boldsymbol{f}_i \cdot \boldsymbol{x}}$ is 0, which implies that r is a function in $C^\infty(\mathbb{R}^n)$ such that $r(\alpha) = 0$, $\forall \alpha \in \mathbb{N}^n$.

It remains to prove that the inverse systems spanned by $\omega_i(\boldsymbol{y}) = \sum_{\alpha \in A_i} g_{i,\alpha} \omega_\alpha(\boldsymbol{y})$ and by $g_i(\boldsymbol{x}) = \sum_{\alpha \in A_i} g_{i,\alpha} \boldsymbol{x}^\alpha$ have the same dimension. The polynomials ω_α are of the form

$$\omega_\alpha(\boldsymbol{y}) = \boldsymbol{y}^\alpha + \sum_{\alpha' \neq \alpha, \alpha' \ll \alpha} \omega_{\alpha,\alpha'} \boldsymbol{y}^{\alpha'},$$

with $\omega_{\alpha,\alpha'} \in \mathbb{Q}$. Let ρ denotes the linear map of $\mathbb{C}[\boldsymbol{y}]$ such that $\rho(\boldsymbol{y}^\alpha) = \omega_\alpha(\boldsymbol{y}) - \boldsymbol{y}^\alpha$. We choose a monomial ordering $\succ$, which is a total ordering on the monomials compatible with the multiplication. Then, the initial $\mathrm{in}(\omega_\alpha)$ of ω_α, that is the maximal monomial of the support of ω_α, is $\boldsymbol{y}^\alpha$ since $\boldsymbol{y}^\alpha \succ \mathrm{in}(\rho(\boldsymbol{y}^\alpha))$. As the support of ω_α is in $\{\alpha', \alpha' \ll \alpha\}$, the support of $\partial^\beta \omega_\alpha$ ($\beta \in \mathbb{N}^n$) is $\{\alpha', \alpha' \ll \alpha - \beta\}$ and the initial of $\partial^\beta \omega_\alpha$ is $\partial^\beta(\boldsymbol{x}^\alpha)$. By linearity, for any $g \in \mathbb{C}[\boldsymbol{y}]$, we have $\mathrm{in}(g) \succ \mathrm{in}(\rho(g))$. We deduce that

$$\omega_i(\boldsymbol{y}) = \sum_{\alpha \in A_i} g_{i,\alpha} \omega_\alpha(\boldsymbol{y}) = \sum_{\alpha \in A_i} g_{i,\alpha} (\boldsymbol{y}^\alpha + \rho(\boldsymbol{y}^\alpha)) = g_i(\boldsymbol{y}) + \rho(g_i)$$

and the initial $\mathrm{in}(\partial^\beta \omega_i)$ is also the initial of $\partial^\beta g_i$ ($\beta \in \mathbb{N}^n$). Therefore the initial of the vector space spanned by $\omega_i(\boldsymbol{y}) = g_i(\boldsymbol{y}) + \rho(g_i)$ and all its derivatives coincides with the vector space spanned by the initial of $\omega_i(\boldsymbol{y}) = g_i(\boldsymbol{y})$ and all its derivatives. Therefore, the two vector spaces have the same dimension. This concludes the proof. □

Remark 5.4.2 *Instead of a shift by* 1 *and the generating series of* h *computed on the unitary grid* $\mathbb{N}^n$, *one can consider the shift* $S_j(h) = h\left(x_1, \dots, x_{j-1}, x_j + \frac{1}{T_j}, x_{j+1}, \dots, x_n\right)$ *for* $T_j \in \mathbb{R}_+$ *and the generating series of the sequence* $\left(h\left(\frac{\alpha_1}{T_1}, \dots, \frac{\alpha_n}{T_n}\right)\right)_{\alpha \in \mathbb{N}^n}$. *The previous results apply directly, replacing the function* h *by* $h_T : (x_1, \dots, x_n) \mapsto h\left(\frac{x_1}{T_1}, \dots, \frac{x_n}{T_n}\right)$ *where* $T = (T_1, \dots, T_n)$.

Remark 5.4.3 *Using Lemma 2.2.4, we check that the map $h \in \mathscr{P}ol\mathscr{E}xp \mapsto \sigma_h \in \mathbb{C}[[\mathbf{y}]]$ is injective and the regularity condition is satisfied on $\mathscr{P}ol\mathscr{E}xp$. Thus, in Theorem 5.4.1 if $h \in \mathscr{P}ol\mathscr{E}xp$ then we must have $r(\mathbf{x}) = 0$.*

Remark 5.4.4 *By applying Algorithm 4.6.2 to the sequence of evaluations of a function $h \in \mathscr{P}ol\mathscr{E}xp$ on the (first) points of a regular grid in $\mathbb{R}^n$, we obtain a method to decompose functions in $\in \mathscr{P}ol\mathscr{E}xp$ as a sum of products of polynomials by exponentials.*

5.5 Sparse interpolation

For $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ and $\mathbf{x} \in \mathbb{C}^n$, we denote $\log^{\beta} \mathbf{x} = \prod_{i=1}^{n} (\log(x_i))^{\beta_i}$ where $\log(x)$ is the principal value of the complex logarithm $\mathbb{C} \setminus \{0\}$. Let

$$\mathscr{P}ol\mathscr{L}og(x_1, \dots, x_n) = \left\{ \sum_{\alpha,\beta} p_{\alpha,\beta} \mathbf{x}^{\alpha} \log^{\beta}(\mathbf{x}), p_{\alpha,\beta} \in \mathbb{C} \right\}$$

be the set of functions, which are the sum of products of polynomials in $\mathbf{x}$ and polynomials in $\log(\mathbf{x})$.

For $h = \sum_{\alpha,\beta} h_{\alpha,\beta} \mathbf{x}^{\alpha} \log^{\beta}(\mathbf{x}) \in \mathscr{P}ol\mathscr{L}og(\mathbf{x})$, we denote by $\varepsilon(h)$ the set of exponents $\alpha \in \mathbb{N}^n$ such that $h_{\alpha,\beta} \neq 0$.

The sparse interpolation problem consists in computing the decomposition of a function p of $\mathscr{P}ol\mathscr{L}og(\mathbf{x})$ as a sum of terms of the form $p_{\alpha,\beta} \mathbf{x}^{\alpha} \log^{\beta}(\mathbf{x})$ from the values of p. We apply the construction introduced in Section 5.1 with

- $\mathscr{F} = \mathscr{P}ol\mathscr{L}og(\mathbf{x})$,
- $S_j : h(x_1, \dots, x_n) \mapsto h(x_1, \dots, x_{j-1}, \lambda_j x_j, x_{j+1}, \dots, x_n)$ the scaling operator of x_j by $\lambda_j \in \mathbb{C}$,
- $\Delta : h(x_1, \dots, x_n) \mapsto \Delta[h] = h(1, \dots, 1)$ the evaluation at $\mathbf{1} = (1, \dots, 1)$.

We easily check that

- the operators S_j are commuting,
- for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, the monomial $\mathbf{x}^{\alpha}$ is an eigenfunction of S_j: $S_j(\mathbf{x}^{\alpha}) = \lambda_j^{\alpha_j} \mathbf{x}^{\alpha}$.
- for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, $\mathbf{x}^{\alpha} \log^{\beta}(\mathbf{x})$ is a generalized eigenfunction of S_j:

$$S_j(\mathbf{x}^{\alpha} \log^{\beta}(\mathbf{x})) = \sum_{0 \leqslant \beta' \leqslant \beta_j} \lambda_j^{\alpha_j} \binom{\beta_j}{\beta'} \log^{\beta_j - \beta'} \lambda_j \log^{\beta'}(x_j) \mathbf{x}^{\alpha} \prod_{k \neq j} \log^{\beta_k}(x_k).$$

More generally, for $\gamma \in \mathbb{N}^n$, we have

$$\begin{aligned} S^\gamma(\boldsymbol{x}^\alpha \log^\beta(\boldsymbol{x})) &= \left(\prod_{i=1}^n (\lambda_i^{\gamma_i} x_i)^{\alpha_i}\right)\left(\prod_{i=1}^n (\gamma_i \log(\lambda_i) + \log(x_i))^{\beta_i}\right) \\ &= \xi^\gamma \boldsymbol{x}^\alpha \left(\sum_{\beta' \ll \beta} \binom{\beta}{\beta'} \gamma^{\beta'} \log^{\beta'}(\boldsymbol{\lambda}) \log^{\beta-\beta'}(\boldsymbol{x})\right) \end{aligned}$$

where $\xi = (\lambda_1^{\alpha_1}, \dots, \lambda_n^{\alpha_n})$. We deduce that,

$$\Delta[S^\gamma(\boldsymbol{x}^\alpha \log^\beta(\boldsymbol{x}))] = \xi^\gamma \gamma^\beta \log^\beta(\boldsymbol{\lambda}). \tag{5.7}$$

Theorem 5.5.1 *Let $h \in \mathscr{P}ol\mathscr{L}og(\boldsymbol{x})$. For $\lambda_1, \dots, \lambda_n \in \mathbb{C}$, the generating series $\sigma_h = \sum_{\gamma \in \mathbb{N}^n} h(\lambda_1^{\gamma_1}, \dots, \lambda_n^{\gamma_n}) \frac{\boldsymbol{y}^\gamma}{\gamma!}$ of h is of the form*

$$\sigma_h(\boldsymbol{y}) = \sum_{i=1}^{r'} \omega_i(\boldsymbol{y}) \boldsymbol{e}_{\xi_i}(\boldsymbol{y})$$

with

- $\varepsilon(h) = \{\alpha_1, \dots, \alpha_{r'}\}$,
- $\xi_i = (\lambda_1^{\alpha_{i,1}}, \dots, \lambda_n^{\alpha_{i,n}}) \in \mathbb{C}^n$,
- $\omega_i(\boldsymbol{y}) = \sum_{\beta \in B_i} \omega_{i,\beta} \boldsymbol{y}^\beta \in \mathbb{C}[\boldsymbol{y}]$.

If moreover $\lambda_i \neq 1$ and the points $\xi_i = (\lambda_1^{\alpha_{i,1}}, \dots, \lambda_n^{\alpha_{i,n}})$, $\alpha_i \in \varepsilon(h)$ are distinct, then $h = \sum_{i=1}^{r'} \sum_{\beta \in B_i} \omega_{i,\beta} \boldsymbol{x}^{\alpha_i} \log^\beta(\boldsymbol{x})$.

Proof. Let $\alpha, \beta \in \mathbb{N}^n$. As $\boldsymbol{x}^\alpha$ is an eigenfunction of the operators S_j, its generating series associated to $\boldsymbol{x}^\alpha$ is $\boldsymbol{e}_\xi(\boldsymbol{y})$ where $\xi = (\lambda_1^{\alpha_1}, \dots, \lambda_n^{\alpha_n})$. From the relations (5.5) and (5.6), we deduce that the generating series of $\boldsymbol{x}^\alpha \log^\beta(\boldsymbol{x})$ is

$$\sigma_{\boldsymbol{x}^\alpha \log^\beta(\boldsymbol{x})} = \log^\beta(\boldsymbol{\lambda}) \sum_{\gamma \in \mathbb{N}^n} \gamma^\beta \xi^\gamma \frac{\boldsymbol{y}^\gamma}{\gamma!} = \log^\beta(\boldsymbol{\lambda}) \omega_\beta(\boldsymbol{y}) \boldsymbol{e}_\xi(\boldsymbol{y})$$

where $\omega_\beta(\boldsymbol{y})$ is the polynomial obtained from the expansion of $\boldsymbol{y}^\beta$ in terms of the Macaulay binomial polynomials $b_\alpha(\boldsymbol{y})$. As in Section 5.4, this shows that the completeness property is satisfied.

If $h = \sum_{i=1}^{r'} \sum_{\beta \in B_i} h_{i,\beta} \boldsymbol{x}^{\alpha_i} \log^\beta(\boldsymbol{x})$, $\lambda_i \neq 1$ and the points $\xi_i = (\lambda_1^{\alpha_{i,1}}, \dots, \lambda_n^{\alpha_{i,n}})$ are distinct, then

$$\sigma_h = \sum_{i=1}^{r'} \left(\sum_{\beta \in B_i} h_{i,\beta} \log^\beta(\boldsymbol{\lambda}) \omega_\beta(\boldsymbol{y})\right) \boldsymbol{e}_{\xi_i}(\boldsymbol{y}) = \sum_{i=1}^{r'} \omega_i(\boldsymbol{y}) \boldsymbol{e}_{\xi_i}(\boldsymbol{y})$$

with $\xi_i = (\lambda_1^{\alpha_{i,1}}, \ldots, \lambda_n^{\alpha_{i,n}})$ and $\omega_i(\boldsymbol{y}) \in \mathbb{C}[\boldsymbol{x}]$. By Lemma 2.2.4 and the linear independency of the polynomials ω_β, we deduce that the coefficients $h_{i,\beta}$ are uniquely determined from the coefficients of the decomposition of $\omega_i(\boldsymbol{y})$ in terms of the Macaulay binomial polynomials ω_β, since $\log^\beta(\boldsymbol{\lambda}) \neq 0$. □

This result leads to a new method to decompose an element $h \in \mathscr{P}ol\mathscr{L}og(\boldsymbol{x})$ with an exponent set $\varepsilon(h) \subset A \subset \mathbb{N}^n$. By choosing $\lambda_1, \ldots, \lambda_n \in \mathbb{C} \setminus \{1\}$ such that the points $(\lambda_1^\alpha, \ldots, \lambda_n^\alpha)$ for $\alpha \in A$ are distinct and by computing the decomposition of the generating series as a polynomial-exponential series $\sum_{i=1}^{r'} \omega_i(\boldsymbol{y}) \boldsymbol{e}_{\xi_i}(\boldsymbol{y})$ (Algorithm 4.6.2), we deduce the exponents $\alpha_i = (\log_{\lambda_1}(\xi_{i,1}), \ldots, \log_{\lambda_n}(\xi_{i,n}))$ and the coefficients $h_{i,\beta}$ in the decomposition $h = \sum_{i=1}^{r'} \sum_{\beta \in B_i} h_{i,\beta} \boldsymbol{x}^{\alpha_i} \log^\beta(\boldsymbol{x})$ from the weight polynomials $\omega_i(\boldsymbol{y})$.

This method generalizes the sparse interpolation methods of [BOT88], [Zip79], [GLL09], where a single operator $S : h(x_1, \ldots, x_n) \mapsto h(\lambda_1 x_1, \ldots, \lambda_n x_n)$ is used for some $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and where only polynomial functions are considered. The monomials $\boldsymbol{x}^\alpha$ ($\alpha \in \mathbb{N}^n$) are eigenfunctions of S for the eigenvalue $\boldsymbol{\lambda}^\alpha = \prod_{i=1}^n \lambda_i^{\alpha_i}$. For $h = \sum_{i=1}^r \omega_i \boldsymbol{x}^{\alpha_i}$, the corresponding univariate generating series σ_h defines an Hankel operator, which kernel is generated by the polynomial $p(x) = \prod_{i=1}^r (x - \boldsymbol{\lambda}^{\alpha_i})$ when $\boldsymbol{\lambda}^{\alpha_1}, \ldots, \boldsymbol{\lambda}^{\alpha_r}$ are distinct. If $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are chosen adequately (for instance distinct prime integers [BOT88], [Zip79] or roots of unity of different orders [GLL09]), the roots of p yield the exponents of $h \in \mathbb{C}[\boldsymbol{x}]$.

The multivariate approach allows to use moments $h(\lambda_1^{\alpha_1}, \ldots, \lambda_n^{\alpha_n})$ with $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ of degree $|\alpha| = \alpha_1 + \cdots + \alpha_n$ less than the degree $2r - 1$ needed in the previous sparse interpolation methods. Sums of products of polynomials and logarithm functions can also be recovered by this method, the logarithm terms corresponding to multiple roots.

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