Category theory
as an abstract programming language

Mohamed Barakat

Universität Siegen

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Joint work with
Markus Lange-Hegermann, Sebastian Gutsche, Florian Heiderich,
Sebastian Posur, Kamal Saleh

Intersection: From concrete algorithms to abstraction

Example (Intersection of subspaces)

Compute the intersection of the two subspaces of $V \coloneqq \mathbb{Q}^{3\times 1}$

$$U_1 \coloneqq \langle (1,2,3),(2,3,4),(0,1,2) \rangle,$$
$$U_2 \coloneqq \langle (1,2,4),(3,2,0) \rangle.$$

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Solution:

$$U_1 \cap U_2 = \langle (1,1,1) \rangle < \mathbb{Q}^{3\times 1}.$$

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Goals

- Describe algorithms to intersect vector subspaces;

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- Describe algorithms to intersect vector subspaces;
- Generalize these algorithms to more general setups.

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- Describe algorithms to intersect vector subspaces;
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Output: $\mathtt{i}_1$ with $U_1 \cap U_2 = \langle \text{rows of the matrix } \mathtt{i}_1 \rangle \leq \mathbb{Q}^{1 \times d}$

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Intersection1 $(\mathtt{u}_1, \mathtt{u}_2)$

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Intersection1 *($\mathtt{u}_1, \mathtt{u}_2$)*

```
m₁ := REF(u₁)                    // row echelon form of u₁
```

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Intersection1 *($\mathtt{u}_1, \mathtt{u}_2$)*

```
m₁ := REF(u₁)        // row echelon form of u₁
m₂ := REF(u₂)
```

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Intersection1 *($\mathtt{u}_1, \mathtt{u}_2$)*

1 $\mathtt{m}_1 \coloneqq \mathtt{REF}(\mathtt{u}_1)$ // row echelon form of $\mathbf{u}_1$

2 $\mathtt{m}_2 \coloneqq \mathtt{REF}(\mathtt{u}_2)$

3 $\left(\begin{array}{c|c} \mathtt{n}_1 & \mathtt{n}_2 \end{array}\right) \coloneqq \mathtt{LeftNullSpace}\left(\left(\begin{array}{c} \mathtt{m}_1 \\ \hline \mathtt{m}_2 \end{array}\right)\right)$

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3 $\left(\begin{array}{c|c} \mathtt{n}_1 & \mathtt{n}_2 \end{array} \right) := \mathtt{LeftNullSpace}\left(\begin{pmatrix} \mathtt{m}_1 \\ \hline \mathtt{m}_2 \end{pmatrix}\right)$

4 $\mathtt{i}_1 := \mathtt{MatMul}(\mathtt{n}_1, \mathtt{m}_1) := \mathtt{n}_1 \mathtt{m}_1$

Algorithm 1 to intersect two vector subspaces

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$U_1 \coloneqq \langle$rows of the matrix $\mathtt{u}_1\rangle, U_2 \coloneqq \langle$rows of the matrix $\mathtt{u}_2\rangle$.

Output: $\mathtt{i}_1$ with $U_1 \cap U_2 = \langle$rows of the matrix $\mathtt{i}_1\rangle \leq \mathbb{Q}^{1 \times d}$

Intersection1 *($\mathtt{u}_1, \mathtt{u}_2$)*

$\mathtt{m}_1 \coloneqq \mathtt{REF}(\mathtt{u}_1)$ // row echelon form of $\mathbf{u}_1$

$\mathtt{m}_2 \coloneqq \mathtt{REF}(\mathtt{u}_2)$

$\left(\; \mathtt{n}_1 \; \middle| \; \mathtt{n}_2 \; \right) \coloneqq \mathtt{LeftNullSpace}\left(\begin{pmatrix} \mathtt{m}_1 \\ \hline \mathtt{m}_2 \end{pmatrix}\right)$

$\mathtt{i}_1 \coloneqq \mathtt{MatMul}(\mathtt{n}_1, \mathtt{m}_1) \coloneqq \mathtt{n}_1 \mathtt{m}_1$

return $\mathtt{i}_1$

Algorithm 2: Intersection of vector subspaces

Input: Two stackable matrices $\mathtt{u}_1, \mathtt{u}_2 \in \mathbb{Q}^{? \times d}$
$U_1 \coloneqq \langle$rows of the matrix $\mathtt{u}_1\rangle, U_2 \coloneqq \langle$rows of the matrix $\mathtt{u}_2\rangle$.
Output: $\mathtt{s}_1$ with $U_1 \cap U_2 = \langle$rows of the matrix $\mathtt{s}_1\rangle \leq \mathbb{Q}^{1 \times d}$
Intersection2 *($\mathtt{u}_1, \mathtt{u}_2$)*

Algorithm 2 to intersect two vector subspaces

Algorithm 2: Intersection of vector subspaces

Input: Two stackable matrices $\mathtt{u}_1, \mathtt{u}_2 \in \mathbb{Q}^{? \times d}$
$U_1 := \langle$rows of the matrix $\mathtt{u}_1\rangle, U_2 := \langle$rows of the matrix $\mathtt{u}_2\rangle$.
Output: $\mathtt{s}_1$ with $U_1 \cap U_2 = \langle$rows of the matrix $\mathtt{s}_1\rangle \leq \mathbb{Q}^{1 \times d}$
Intersection2 *($\mathtt{u}_1, \mathtt{u}_2$)*

```
1    e₂ := RightNullSpace(u₂)
```

Algorithm 2: Intersection of vector subspaces

Input: Two stackable matrices $\mathtt{u}_1, \mathtt{u}_2 \in \mathbb{Q}^{? \times d}$
$U_1 := \langle$rows of the matrix $\mathtt{u}_1\rangle, U_2 := \langle$rows of the matrix $\mathtt{u}_2\rangle$.

Output: $\mathtt{s}_1$ with $U_1 \cap U_2 = \langle$rows of the matrix $\mathtt{s}_1\rangle \leq \mathbb{Q}^{1 \times d}$

Intersection2 *($\mathtt{u}_1, \mathtt{u}_2$)*

```
e₂ := RightNullSpace(u₂)
w₁ := MatMul(u₁, e₂) := u₁e₂
```

Algorithm 2: Intersection of vector subspaces

Input: Two stackable matrices $\mathtt{u}_1, \mathtt{u}_2 \in \mathbb{Q}^{? \times d}$
$U_1 \coloneqq \langle$rows of the matrix $\mathtt{u}_1\rangle, U_2 \coloneqq \langle$rows of the matrix $\mathtt{u}_2\rangle$.
Output: $\mathtt{s}_1$ with $U_1 \cap U_2 = \langle$rows of the matrix $\mathtt{s}_1\rangle \leq \mathbb{Q}^{1 \times d}$
Intersection2 *($\mathtt{u}_1, \mathtt{u}_2$)*

1. $\mathtt{e}_2 \coloneqq \mathtt{RightNullSpace}(\mathtt{u}_2)$
2. $\mathtt{w}_1 \coloneqq \mathtt{MatMul}(\mathtt{u}_1, \mathtt{e}_2) \coloneqq \mathtt{u}_1\mathtt{e}_2$
3. $\mathtt{k}_1 \coloneqq \mathtt{LeftNullSpace}(\mathtt{w}_1)$

Algorithm 2 to intersect two vector subspaces

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Output: $\mathtt{s}_1$ with $U_1 \cap U_2 = \langle$rows of the matrix $\mathtt{s}_1\rangle \leq \mathbb{Q}^{1 \times d}$

Intersection2 *($\mathtt{u}_1, \mathtt{u}_2$)*

```
e₂ := RightNullSpace(u₂)
w₁ := MatMul(u₁, e₂) := u₁e₂
k₁ := LeftNullSpace(w₁)
v₁ := MatMul(k₁, u₁) := k₁u₁
```

Algorithm 2 to intersect two vector subspaces

Algorithm 2: Intersection of vector subspaces

Input: Two stackable matrices $\mathtt{u}_1, \mathtt{u}_2 \in \mathbb{Q}^{? \times d}$

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Output: $\mathtt{s}_1$ with $U_1 \cap U_2 = \langle$rows of the matrix $\mathtt{s}_1\rangle \leq \mathbb{Q}^{1 \times d}$

Intersection2 *(*$\mathtt{u}_1, \mathtt{u}_2$*)*

$\mathtt{e}_2 \coloneqq \mathtt{RightNullSpace}(\mathtt{u}_2)$

$\mathtt{w}_1 \coloneqq \mathtt{MatMul}(\mathtt{u}_1, \mathtt{e}_2) \coloneqq \mathtt{u}_1 \mathtt{e}_2$

$\mathtt{k}_1 \coloneqq \mathtt{LeftNullSpace}(\mathtt{w}_1)$

$\mathtt{v}_1 \coloneqq \mathtt{MatMul}(\mathtt{k}_1, \mathtt{u}_1) \coloneqq \mathtt{k}_1 \mathtt{u}_1$

$\mathtt{s}_1 \coloneqq \mathtt{REF}(\mathtt{v}_1)$

Algorithm 2 to intersect two vector subspaces

Algorithm 2: Intersection of vector subspaces

Input: Two stackable matrices $\mathtt{u}_1, \mathtt{u}_2 \in \mathbb{Q}^{? \times d}$
$U_1 \coloneqq \langle$rows of the matrix $\mathtt{u}_1\rangle, U_2 \coloneqq \langle$rows of the matrix $\mathtt{u}_2\rangle$.

Output: $\mathtt{s}_1$ with $U_1 \cap U_2 = \langle$rows of the matrix $\mathtt{s}_1\rangle \leq \mathbb{Q}^{1 \times d}$

Intersection2 *($\mathtt{u}_1, \mathtt{u}_2$)*

$\mathtt{e}_2 \coloneqq \mathtt{RightNullSpace}(\mathtt{u}_2)$
$\mathtt{w}_1 \coloneqq \mathtt{MatMul}(\mathtt{u}_1, \mathtt{e}_2) \coloneqq \mathtt{u}_1 \mathtt{e}_2$
$\mathtt{k}_1 \coloneqq \mathtt{LeftNullSpace}(\mathtt{w}_1)$
$\mathtt{v}_1 \coloneqq \mathtt{MatMul}(\mathtt{k}_1, \mathtt{u}_1) \coloneqq \mathtt{k}_1 \mathtt{u}_1$
$\mathtt{s}_1 \coloneqq \mathtt{REF}(\mathtt{v}_1)$
return $\mathtt{s}_1$

Algorithm 3 to intersect two vector subspaces

Algorithm 3: Intersection of vector subspaces

Input: Two stackable matrices $\mathtt{u}_1, \mathtt{u}_2 \in \mathbb{Q}^{? \times d}$
$U_1 \coloneqq \langle$rows of the matrix $\mathtt{u}_1\rangle, U_2 \coloneqq \langle$rows of the matrix $\mathtt{u}_2\rangle$.
Output: $\mathtt{k}$ with $U_1 \cap U_2 = \langle$rows of the matrix $\mathtt{k}\rangle \leq \mathbb{Q}^{1 \times d}$
Intersection3 *($\mathtt{u}_1, \mathtt{u}_2$)*

Algorithm 3 to intersect two vector subspaces

Algorithm 3: Intersection of vector subspaces

Input: Two stackable matrices $\mathtt{u}_1, \mathtt{u}_2 \in \mathbb{Q}^{? \times d}$
$U_1 := \langle$rows of the matrix $\mathtt{u}_1\rangle, U_2 := \langle$rows of the matrix $\mathtt{u}_2\rangle$.
Output: $\mathtt{k}$ with $U_1 \cap U_2 = \langle$rows of the matrix $\mathtt{k}\rangle \leq \mathbb{Q}^{1 \times d}$
Intersection3 *(*$\mathtt{u}_1, \mathtt{u}_2$*)*

```
e₁ := RightNullSpace(u₁)
```

Algorithm 3 to intersect two vector subspaces

Algorithm 3: Intersection of vector subspaces

Input: Two stackable matrices $\mathtt{u}_1, \mathtt{u}_2 \in \mathbb{Q}^{?\times d}$

$U_1 \coloneqq \langle$rows of the matrix $\mathtt{u}_1\rangle, U_2 \coloneqq \langle$rows of the matrix $\mathtt{u}_2\rangle$.

Output: $\mathtt{k}$ with $U_1 \cap U_2 = \langle$rows of the matrix $\mathtt{k}\rangle \leq \mathbb{Q}^{1\times d}$

Intersection3 *($\mathtt{u}_1, \mathtt{u}_2$)*

```
e1 := RightNullSpace(u1)
e2 := RightNullSpace(u2)
```

Algorithm 3 to intersect two vector subspaces

Algorithm 3: Intersection of vector subspaces

Input: Two stackable matrices $\mathtt{u}_1, \mathtt{u}_2 \in \mathbb{Q}^{? \times d}$
$U_1 \coloneqq \langle$rows of the matrix $\mathtt{u}_1\rangle, U_2 \coloneqq \langle$rows of the matrix $\mathtt{u}_2\rangle$.
Output: $\mathtt{k}$ with $U_1 \cap U_2 = \langle$rows of the matrix $\mathtt{k}\rangle \leq \mathbb{Q}^{1 \times d}$

Intersection3 *($\mathtt{u}_1, \mathtt{u}_2$)*

```
e1 := RightNullSpace(u1)
e2 := RightNullSpace(u2)
a := Augment(e1, e2)
```

Algorithm 3 to intersect two vector subspaces

Algorithm 3: Intersection of vector subspaces

Input: Two stackable matrices $\mathtt{u}_1, \mathtt{u}_2 \in \mathbb{Q}^{? \times d}$
$U_1 \coloneqq \langle$rows of the matrix $\mathtt{u}_1\rangle, U_2 \coloneqq \langle$rows of the matrix $\mathtt{u}_2\rangle$.
Output: $\mathtt{k}$ with $U_1 \cap U_2 = \langle$rows of the matrix $\mathtt{k}\rangle \leq \mathbb{Q}^{1 \times d}$

Intersection3 *($\mathtt{u}_1, \mathtt{u}_2$)*

$\mathtt{e}_1 \coloneqq$ RightNullSpace$(\mathtt{u}_1)$
$\mathtt{e}_2 \coloneqq$ RightNullSpace$(\mathtt{u}_2)$
$\mathtt{a} \coloneqq$ Augment$(\mathtt{e}_1, \mathtt{e}_2)$
$\mathtt{k} \coloneqq$ LeftNullSpace$(\mathtt{a})$

Algorithm 3 to intersect two vector subspaces

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Input: Two stackable matrices $\mathtt{u}_1, \mathtt{u}_2 \in \mathbb{Q}^{? \times d}$
$U_1 \coloneqq \langle$rows of the matrix $\mathtt{u}_1\rangle, U_2 \coloneqq \langle$rows of the matrix $\mathtt{u}_2\rangle$.
Output: $\mathtt{k}$ with $U_1 \cap U_2 = \langle$rows of the matrix $\mathtt{k}\rangle \leq \mathbb{Q}^{1 \times d}$

```
Intersection3 (u1, u2)
    e1 := RightNullSpace(u1)
    e2 := RightNullSpace(u2)
    a := Augment(e1, e2)
    k := LeftNullSpace(a)
    return k
```

Algorithm 4 = 3' to intersect two vector subspaces

Algorithm 4: Intersection of vector subspaces

Input: Two stackable matrices $\mathtt{u}_1, \mathtt{u}_2 \in \mathbb{Q}^{? \times d}$
$U_1 := \langle \text{rows of the matrix } \mathtt{u}_1 \rangle, U_2 := \langle \text{rows of the matrix } \mathtt{u}_2 \rangle$.
Output: $\mathtt{z}_0$ with $U_1 \cap U_2 = \langle \text{rows of the matrix } \mathtt{z}_0 \rangle \leq \mathbb{Q}^{1 \times d}$
Intersection4 *($\mathtt{u}_1, \mathtt{u}_2$)*

Algorithm 4: Intersection of vector subspaces

Input: Two stackable matrices $\mathtt{u}_1, \mathtt{u}_2 \in \mathbb{Q}^{? \times d}$
$U_1 \coloneqq \langle$rows of the matrix $\mathtt{u}_1\rangle, U_2 \coloneqq \langle$rows of the matrix $\mathtt{u}_2\rangle$.
Output: $\mathtt{z}_0$ with $U_1 \cap U_2 = \langle$rows of the matrix $\mathtt{z}_0\rangle \leq \mathbb{Q}^{1 \times d}$

Intersection4 $(\mathtt{u}_1, \mathtt{u}_2)$

```
d := NrColumns(u1)
```

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Input: Two stackable matrices $\mathtt{u}_1, \mathtt{u}_2 \in \mathbb{Q}^{? \times d}$
$U_1 \coloneqq \langle$rows of the matrix $\mathtt{u}_1\rangle, U_2 \coloneqq \langle$rows of the matrix $\mathtt{u}_2\rangle$.
Output: $\mathtt{z}_0$ with $U_1 \cap U_2 = \langle$rows of the matrix $\mathtt{z}_0\rangle \leq \mathbb{Q}^{1 \times d}$

Intersection4 $(\mathtt{u}_1, \mathtt{u}_2)$

```
d := NrColumns(u1)
i := IdentityMat(d, Q)
```

Algorithm 4: Intersection of vector subspaces

Input: Two stackable matrices $\mathtt{u}_1, \mathtt{u}_2 \in \mathbb{Q}^{?\times d}$

$U_1 \coloneqq \langle$rows of the matrix $\mathtt{u}_1\rangle, U_2 \coloneqq \langle$rows of the matrix $\mathtt{u}_2\rangle$.

Output: $\mathtt{z}_0$ with $U_1 \cap U_2 = \langle$rows of the matrix $\mathtt{z}_0\rangle \leq \mathbb{Q}^{1\times d}$

Intersection4 *($\mathtt{u}_1, \mathtt{u}_2$)*

$\mathtt{d} \coloneqq \texttt{NrColumns}(\mathtt{u}_1)$

$\mathtt{i} \coloneqq \texttt{IdentityMat}(\mathtt{d}, \mathbb{Q})$

$$\mathtt{p} \coloneqq \texttt{Stack}(\texttt{Augment}(\mathtt{i},\mathtt{i}), \texttt{Diag}(\mathtt{u}_1,\mathtt{u}_2)) \coloneqq \begin{pmatrix} 1 & 1 \\ \hline \mathtt{u}_1 & 0 \\ 0 & \mathtt{u}_2 \end{pmatrix}$$

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$$\mathtt{p} \coloneqq \mathtt{Stack}(\mathtt{Augment}(\mathtt{i}, \mathtt{i}), \mathtt{Diag}(\mathtt{u}_1, \mathtt{u}_2)) \coloneqq \left(\begin{array}{cc} 1 & 1 \\ \hline \mathtt{u}_1 & 0 \\ 0 & \mathtt{u}_2 \end{array}\right)$$

$$\left(\begin{array}{c|cc} \mathtt{z}_0 & \mathtt{z}_1 & \mathtt{z}_2 \end{array} \right) \coloneqq \mathtt{LeftNullSpace}(\mathtt{p})$$

Algorithm 4 = 3' to intersect two vector subspaces

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$U_1 := \langle$rows of the matrix $\mathtt{u}_1\rangle, U_2 := \langle$rows of the matrix $\mathtt{u}_2\rangle$.
Output: $\mathtt{z}_0$ with $U_1 \cap U_2 = \langle$rows of the matrix $\mathtt{z}_0\rangle \leq \mathbb{Q}^{1 \times d}$
Intersection4 *($\mathtt{u}_1, \mathtt{u}_2$)*

$\mathtt{d} := \mathtt{NrColumns}(\mathtt{u}_1)$

$\mathtt{i} := \mathtt{IdentityMat}(\mathtt{d}, \mathbb{Q})$

$$\mathtt{p} := \mathtt{Stack}(\mathtt{Augment}(\mathtt{i}, \mathtt{i}), \mathtt{Diag}(\mathtt{u}_1, \mathtt{u}_2)) := \left(\begin{array}{cc} 1 & 1 \\ \hline \mathtt{u}_1 & 0 \\ 0 & \mathtt{u}_2 \end{array}\right)$$

$\left(\begin{array}{c|cc} \mathtt{z}_0 & \mathtt{z}_1 & \mathtt{z}_2 \end{array}\right) := \mathtt{LeftNullSpace}(\mathtt{p})$

return $\mathtt{z}_0$

Our goals were ...

Goals

- Describe algorithms to intersect vector subspaces;
- Generalize these algorithms to more general setups.

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Main idea

Describe the subspaces $U_1, U_2 \leq V$ as the image of linear maps u_1, u_2 defined by the matrices $\mathtt{u}_1, \mathtt{u}_2$, respectively:

$$u_1 : \mathbb{Q}^{g_1 \times 1} \xrightarrow{\mathtt{u}_1} \mathbb{Q}^{d \times 1},$$
$$u_2 : \mathbb{Q}^{g_2 \times 1} \xrightarrow{\mathtt{u}_2} \mathbb{Q}^{d \times 1}.$$

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Main idea

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$$u_1 : \mathbb{Q}^{g_1 \times 1} \xrightarrow{\mathtt{u}_1} \mathbb{Q}^{d \times 1},$$
$$u_2 : \mathbb{Q}^{g_2 \times 1} \xrightarrow{\mathtt{u}_2} \mathbb{Q}^{d \times 1}.$$

Vector spaces together with their linear maps form a **category**.

Data structures and algorithms for a category

A **quiver** (directed multi-graph) $\mathcal{C}$ consists of

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- a class of objects $\mathcal{C}_0$;

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A **quiver** (directed multi-graph) $\mathcal{C}$ consists of

- a class of objects $\mathcal{C}_0$;
- a class of morphisms $\mathcal{C}_1$

- two structure maps:

(1,2) source and target $s, t : \mathcal{C}_1 \to \mathcal{C}_0$;

Data structures and algorithms for a category

A **quiver** (directed multi-graph) $\mathcal{C}$ consists of

- a class of objects $\mathcal{C}_0$;
- a class of morphisms $\mathcal{C}_1 := \dot{\bigcup}_{M,N\in\mathcal{C}_0} \underbrace{\operatorname{Hom}_{\mathcal{C}}(M,N)}_{(s\times t)^{-1}(M,N)}$;
- two structure maps:

(1,2) source and target $s, t : \mathcal{C}_1 \to \mathcal{C}_0$;

Data structures and algorithms for a category

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Categories up to equivalence emphasize morphisms and treat objects merely as place holders for sources and targets.

Further definitions and examples

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- **small** if both $\mathcal{C}_0$ and $\mathcal{C}_1$ are sets;

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Example

(associative unital) ring ≡ ringoid on one object

Further doctrines: k-linear categories

Let k be commutative unital ring.

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In particular, $\mathbf{CatClosure}$ *invents* the word calculus.

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- A finite **product** of objects in a category $\mathcal{C}$

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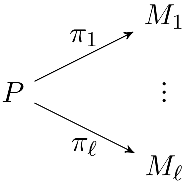

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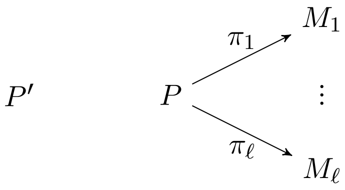

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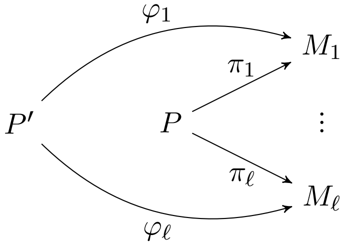

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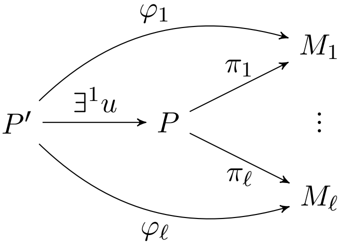

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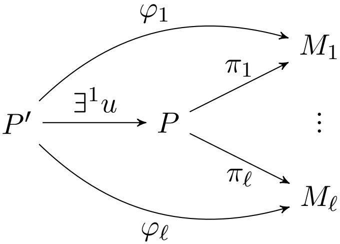

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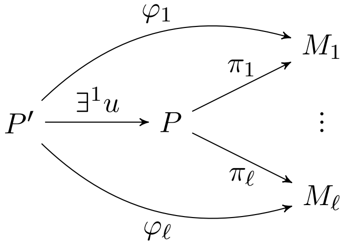

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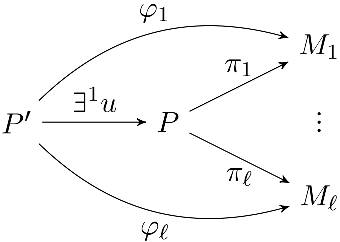

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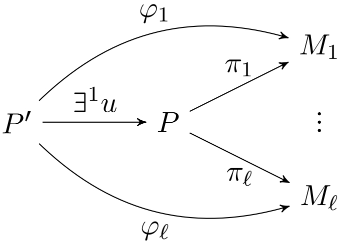

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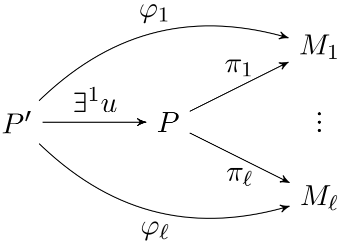

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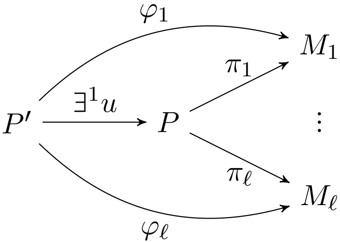

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Q:

What are the initial and terminal objects in $\mathbf{SkeletalFintSets}$?

Additive and k-additive categories

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$$\coloneqq \dot{\bigcup}_{g,g' \in \mathbb{N}} k^{g \times g'} \coloneqq \begin{cases} \text{Obj:} & g, g', \ldots \in \mathbb{N}, \\ \text{Mor:} & \mathtt{A} \in k^{g \times g'}, \ldots, \end{cases}$$

with the obvious 4 algorithms for $s, t, 1, \mu$.

Additive and k-additive categories

Definition

- A **biproduct** of objects in $\mathcal{C}$ is a product and a coproduct simultaneously: We write $M_1 \oplus \cdots \oplus M_\ell$.
- A pre-additive category is **additive** if it admits biproducts.

Example (One more category constructor)

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In particular, $\mathbf{AdditiveClosure}$ *invents* matrix calculus.

How to model the category of finite dimensional vector spaces up to equivalence?

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Example

Let k be a field.

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$k\text{-}\mathbf{vec} \simeq k\text{-}\mathbf{mat}$ has much more structure.

ABELian categories as a specification

An ABELian category is a category in which we can do a very general form of linear algebra.

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Definition

A category is called **computable** ABELian if all disjunctions ($\vee$) and all existential quantifiers ($\exists$) in the axioms of an ABELian category are realized by algorithms.

The “hidden” existential quantifiers of “kernels”

Example

Let $\varphi : M \to N$ be a morphism in $\mathcal{A}$.

$$M \xrightarrow{\varphi} N$$

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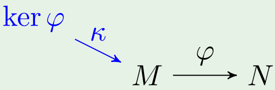

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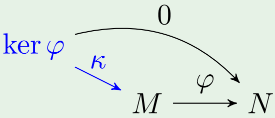

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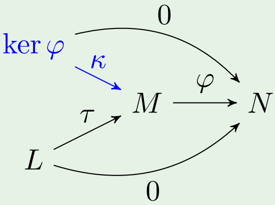

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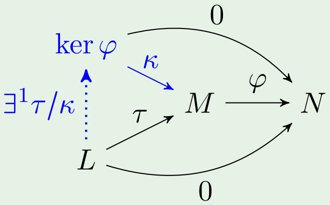

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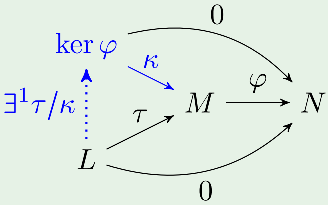

So $\mathcal{A}$ is a computational context with *many* basic algorithms.

Categorical algorithms of k-$\mathbf{mat}$

Proposition

k-$\mathbf{mat}$ is a computable Abelian category.

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Proof.

- Objects m, n are natural numbers in $\mathbb{N}$

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- $s(\varphi) = \texttt{Source}(\varphi) := \texttt{NrRows}(\varphi)$
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- $1_m = \texttt{IdentityMorphism}(m)$

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Categorical algorithms of $k\text{-}\mathbf{mat}$

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- $\mathtt{DirectSum}(m, n)$

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- $\texttt{UniversalMorphismIntoDirectSum}(\varphi, \psi)$

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Categorical algorithms of k-$\mathbf{mat}$

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Categorical algorithms of k-$\mathbf{mat}$

Proposition

k-$\mathbf{mat}$ is a computable Abelian category.

Proof. (continued)

- $\mathtt{KernelObject}(\varphi)$

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Proof. (continued)

- $\texttt{KernelObject}(\varphi) \coloneqq \texttt{NrRows}(\varphi) - \texttt{Rank}(\varphi)$

Categorical algorithms of k-$\mathbf{mat}$

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Proof. (continued)

- $\texttt{KernelObject}(\varphi) \coloneqq \texttt{NrRows}(\varphi) - \texttt{Rank}(\varphi)$
- $\texttt{KernelEmbedding}(\varphi)$

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Proof. (continued)

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- $\mathtt{CokernelObject}(\varphi)$

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- $\texttt{CokernelObject}(\varphi) \coloneqq \texttt{NrColumns}(\varphi) - \texttt{Rank}(\varphi)$
- $\texttt{CokernelProjection}(\varphi) \coloneqq \texttt{CEF}(\texttt{RightNullSpace}(\varphi))$

Categorical algorithms of k-$\mathbf{mat}$

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Categorical algorithms of k-$\mathbf{mat}$

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Proof. (continued)

- $\texttt{KernelObject}(\varphi) := \texttt{NrRows}(\varphi) - \texttt{Rank}(\varphi)$
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- $\texttt{CokernelColift}(\varphi, \tau) :=$
  $\texttt{Sol}(\texttt{CEF}(\texttt{RightNullSpace}(\varphi)) \cdot \chi = \tau)$
- $\texttt{LiftAlongMonomorphism}(\kappa, \tau)$

Categorical algorithms of k-$\mathbf{mat}$

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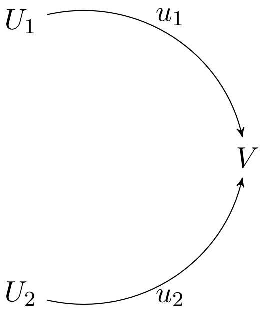
- $\mathtt{CokernelColift}(\varphi, \tau) \coloneqq$ $\mathtt{Sol}(\mathtt{CEF}(\mathtt{RightNullSpace}(\varphi)) \cdot \chi = \tau)$
- $\mathtt{LiftAlongMonomorphism}(\kappa, \tau) \coloneqq \mathtt{Sol}(\chi \cdot \kappa = \tau)$
- $\mathtt{ColiftAlongEpimorphism}(\varepsilon, \tau) \coloneqq \mathtt{Sol}(\varepsilon \cdot \chi = \tau)$ $\square$

Intersection in Abelian categories



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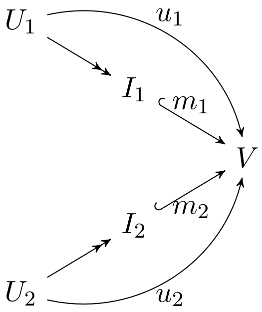

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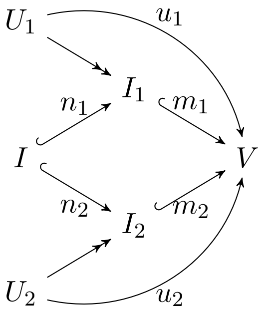

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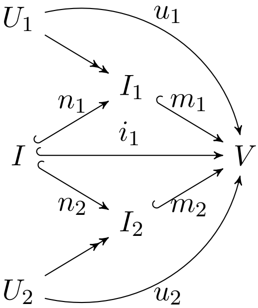

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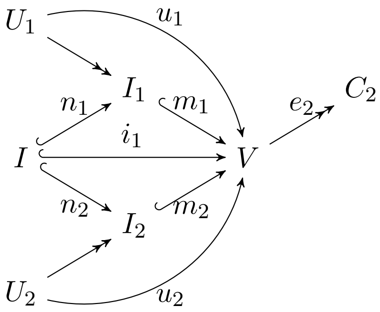

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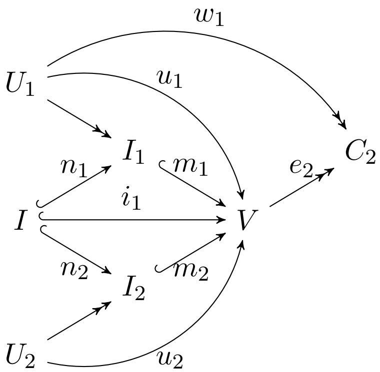

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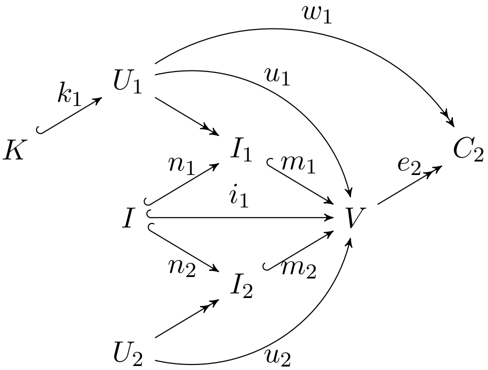

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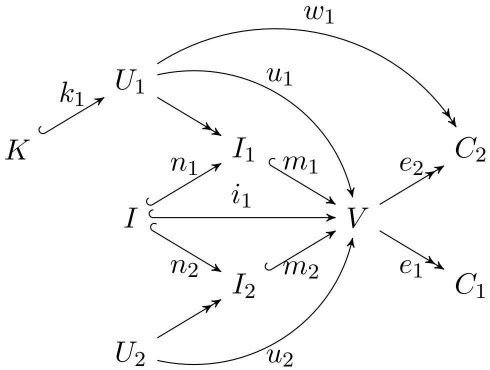

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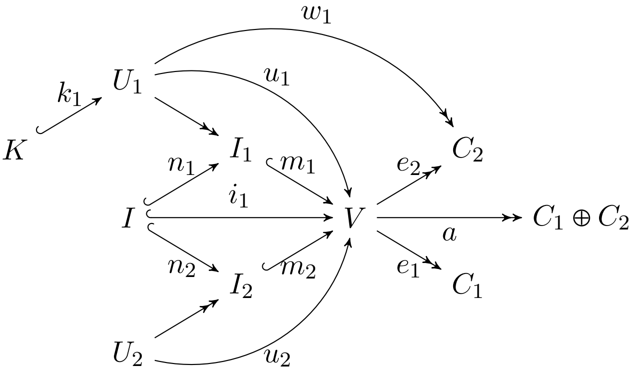

A computable model for R-fdmod

Let k be a field.

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Proposition (GAP-package `FunctorCategories`)

If $\mathcal{B}$ is a finitely presented k-linear category (k-algebroid) and $\mathcal{A}$ is computable ABELian over k, then the functor category

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Corollary

Let R be a finitely presented k-algebra (or k-algebroid), then the category of *finite dimensional* R-modules

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What about finitely presented modules?

Computable rings

From now on let R be a ring with 1.

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If R is left computable then the category $\dot{\bigcup}_{g,g' \in \mathbb{N}} R^{g \times g'}$ is computable additive with weak kernels and decidable lifts.

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Now to a computable model for the category of f.p. R-modules:

A computable model for R-fpmod

Freyd construction $\mathbf{Freyd}(\mathbf{P})$

Let $\mathbf{P}$ be an additive category, then a particular ideal quotient

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$\mathbf{Freyd}(\mathbf{AdditiveClosure}(R\text{-}\mathbf{LinClosure}(\mathbf{CatClosure}(\bullet))))$!!

Examples of computable rings

Example (computable rings)

ring	algorithm
a constructive field k	GAUSS
ring of rational integers $\mathbb{Z}$	HERMITE normal form
a univariate polynomial ring $k[x]$	HERMITE normal form
a polynomial ring[a] $R[x_1, \ldots, x_n]$	BUCHBERGER
many noncommutative rings	n.c. BUCHBERGER
$k[x_1, \ldots, x_n]_{\mathfrak{p}}$	~~MORA~~ BUCHBERGER
residue class rings[b]	
...	

[a] R any of the above rings
[b] modulo ideals which are f.g. as left resp. right ideals.

In this context any algorithm to compute a GRÖBNER basis is a substitute for the GAUSS resp. HERMITE normal form algorithm.

Question

Q:

$\mathbf{Freyd}^2(\mathbf{AdditiveClosure}(R\text{-}\mathbf{LinClosure}(\mathbf{CatClosure}(q)..)$?

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Category theory "invents" data structures and calculi

Free instance of a doctrine	Calculus

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Software demo

Thank you

- Mohamed Barakat and Markus Lange-Hegermann, *An axiomatic setup for algorithmic homological algebra and an alternative approach to localization*, J. Algebra Appl. **10** (2011), no. 2, 269–293, (`arXiv:1003.1943`). MR 2795737 (2012f:18022)
- Sebastian Posur, *A constructive approach to Freyd categories*, ArXiv e-prints (2017), (`arXiv:1712.03492`).