Implicit matrix representations via quadratic relations

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December 21, 2018

Parametric forms are widely used as representations of curves in geometric modelling and computer aided design. The procedure of converting a parametric form into an implicit form, namely implicitization, in the case of plane curves had been studied extensively by means of resultant matrices such as Sylvester, denoted by *Syl*, Bézout or Hybrid Bézout matrices [3, 4], Groebner bases [5, 6] and syzygy based matrices, denoted by *Mrep*, [1, 7, 8]. Given a parameterization of general degree d plane curve $\mathcal{C}$

$$\begin{aligned} \mathbb{R} &\to \mathbb{R}^2 \\ s &\mapsto \left(\frac{f_1(s)}{f_0(s)}, \frac{f_2(s)}{f_0(s)}\right), \end{aligned} \tag{1}$$

the implicit equation of $\mathcal{C}$ is an element of the ideal $I := (f_0T_1 - f_1, f_0T_2 - f_2) \in \mathbb{R}[s, T_1, T_2]$. It is known that the ideal I can be generated by p and q where $p,\ q \in I \subset \mathbb{R}[s, T_1, T_2]$ such that $deg_s(p) = \mu_1$, $deg_s(q) = \mu_2$, $deg_{T_1,T_2}(p) = deg_{T_1,T_2}(q) = 1$ and $\mu_1 + \mu_2 = d$. We will refer to the ideal generators p and q as the μ-basis of the ideal I. Without loss of generality, we may assume that $\mu_2 \geq \mu_1$. We have the following table where $\mu_2 = \lceil \frac{d}{2} \rceil$ for a general plane curve of degree d

size of the matrix	type of resultant matrix
$(2d \times 2d)$	$Syl(f_0T_1 - f_1, f_0T_2 - f_2)$,
$(d \times d)$	$Syl(p, q)$,
$(\mu_2 \times \mu_2)$	Bézout and Hybrid Bézout of p, q.

For a parameterization of a degree d general space curve $\mathcal{D}$ in $\mathbb{R}^n$,

$$\begin{aligned} \mathbb{R} &\to \mathbb{R}^n \\ s &\mapsto \left(\frac{f_1(s)}{f_0(s)}, \frac{f_2(s)}{f_0(s)}, \cdots, \frac{f_n(s)}{f_0(s)}\right), \end{aligned} \tag{2}$$

where $f_0, \cdots, f_n$ are linearly independent polynomials. Its μ-basis exists and it consists of n polynomials $p_1, \cdots, p_n$ of degree $\mu_1, \cdots, \mu_n$ respectively with respect to s such that $p_i = p_{i,0}(s) + \sum_{j=1}^{n} p_{i,j}(s)T_j$ with $deg_s(p_{i,j}(s)) \leq \mu_i$ and there exist at least one $k \in \{0, \cdots, n\}$ such that $deg_s(p_{i,k}(s)) = \mu_i$. We may assume that $\mu_n \geq \mu_{n-1} \geq \cdots \geq \mu_1$. It is known that $\mu_1 + \cdots + \mu_n = d$. If the μ-bases $p_1, \cdots, p_n$ is known, by computing the resultants of p_i, p_j with respect to s, where $1 \leq i < j \leq n$, we obtain $\frac{n(n-1)}{2}$ implicit surfaces. The curve $\mathcal{D}$ is contained in the intersection of these implicit surfaces. However, this intersection may contain extraneous points, i.e. points which are not on the parametric curve $\mathcal{D}$, [9]. It is also known that the implicit matrix representation by moving lines *Mrep* has $\mu_n + \mu_{n-1}$ rows, [1].

In this talk, we introduce a new implicit matrix representation, that we denote by *QMrep*, of a rational parametric curve $\mathcal{D}$ in $\mathbb{R}^n$ by means of quadratic relations. We show that it is possible to obtain an implicit matrix representation *QMrep* with quadratic entries having μ_n rows. We recall that for a degree d general parametric curve in $\mathbb{R}^n$, $\mu_n = \lceil \frac{d}{n} \rceil$, [9]. Compared to the *Mrep* of $\mathcal{D}$, [1], *QMrep* of $\mathcal{D}$ has half number of rows.

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