Structured low rank decomposition and completion of multivariate Hankel matrices

Jouhayna Harmouch Houssam Khalil Bernard Mourrain

Given the first moments of a formal power series $\sigma = (\sigma_\alpha)_{|\alpha|\le d}$ where d is a positive integer, we study the decomposition of a multivariate Hankel matrix H_σ as the sum of low rank matrices in correlation with the decomposition of its associated series as the sum of polynomial-exponential series such that $\sigma(\mathbf{y}) = \sum_{i=1}^{r} \omega_i \, \mathbf{e}_{\xi_i}(\mathbf{y})$ where $\xi_i \in \mathbb{C}^n$ are called frequencies and $\omega_i \in \mathbb{C} \setminus \{0\}$ are called weights. Univariate Hankel matrix $H = [\sigma_{i+j}]_{1\le i\le m, 1\le j\le n}$ associated to σ is structured matrix where the moments depend only on the sum of row and column indices (i.e. the elements on the anti-diagonals are equal). Multivariate Hankel matrix associated to σ is structured matrix $H = [\sigma_{\alpha+\beta}]_{\alpha\in\mathbb{N}^n, \beta\in\mathbb{N}^n}$ where the moments depend only the sum of row and column multi-indices. The Hankel matrix is uniquely identified by its moments. The computation of the basis B of $\mathcal{A}_\sigma$ and its dual B^* of $\mathcal{A}_\sigma^*$ is closely related to the decomposition of an invertible submatrix of H_σ in them. We use the Singular Value Decomposition to compute the bases B and B^* which satisfy the invertibility of Hankel submatrix. We investigate eigen-structure properties of multiplication operators and their transpose to compute weights and frequencies. We use some shifted sub-matrices of Hankel matrix to compute the multiplications operators defined on B and their transpose B^*.

We describe a direct method [4] for the decomposition of multivariate Hankel matrix, based on simple linear algebra tools. The decomposition algorithm applies to matrices of low enough rank. We follow the approach in [6] but directly apply numerically stable linear algebra tools on submatrices of the Hankel matrices to recover the decomposition. The algorithm does not require the solution of polynomial equations. It is connected to the Prony method which constructs sum of exponentials from equally spaced moments by computing a polynomial in the kernel of a Hankel matrix and by deducing the decomposition from the roots of the polynomial [7].

A symmetric tensor T is a tensor whose components stay invariant by any permutation of indices. We show the correlation between the dual of a tensor T^*, formal power series and the Hankel matrices associated to them. We adapt the method of decomposition of Hankel matrices of low rank described in [4] to a decomposition of symmetric tensors method which is based on the decomposition of a formal power series as a weighted sum of exponential described in [6].

To decompose a tensor associated to a real mathematical model, we sometimes do not have enough number of moments to apply the direct method of decomposition. In some applications of tensor decomposition problem [5], we compute the missing data in order to satisfy the constraints of the direct decomposition. The problem of recovering of an unknown Hankel matrix from a small number of entries appears in many applications such that the LDA model where the rank of the matrix is bigger than the dimension [3]. The completion problem is a rank minimization problem which is NP hard and hard to solve [2].

The Rank minimization problem of a matrix denoted by RMP is NP-hard to solve. We use relaxation techniques called Semi Definite programming and denoted by SDP to minimize the trace of the matrix when it is semi definite positive or the nuclear norm when it is not semi definite positive neither symmetric. We adapt these two heuristics to the case of completion of Hankel matrix with some known entries. Given a tensor T_0 of degree 3 in two variables x_1, x_2, we associate the Hankel matrix H_0 to it and we minimize the nuclear norm of H_0 using SDP to complete it. To test the Hankel structure of output matrix, we apply the decomposition method, we compute weights and points and we recover the moments of H_f. We deduce that the moments of H_0 and H_f are almost the same.

We also adapt the singular value thresholding SVT algorithm which is a type of Lagrangian algorithm to minimize the nuclear norm of the matrix when it is Hankel with small number of entries. This algorithm converges if the threshold is big enough even when the matrix is of big dimensions [1]. We adapt the singular value thresholding SVT algorithm to minimize the nuclear norm of the Hankel matrix with few number of known entries. This iterative algorithm produces a sequence of matrices (X^k, Y^k) and at each step performs a soft thresholding algorithm operation on the singular values of the matrix Y^k which consists only of keeping the singular values which are bigger than threshold and moving them towards zero. The choice of a big threshold reduces the storage space at each iteration and the computational cost. It is proved that the iterates of the algorithm converge under the condition of big threshold. We adapt the nuclear norm minimization problem to the Hankel case. We compute the linear operator $\mathcal{A}(X)$ which describes the constraints of the low nuclear norm minimization problem in the case of Hankel matrix with a fixed number of known entries and its transpose $\mathcal{A}^\intercal(y)$. The SVT algorithm converges to a Hankel matrix which maintains the values of known entries and the condition of low rank. In some cases the completion does not provide a Hankel matrix, we call the newton iteration to minimize the distance between the matrix and its representation as a Hankel matrix.

Références

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