A computer-assisted proof for a new lower bound on $\mathcal{H}(4)$ in Hilbert's sixteenth problem

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December 20, 2018

We provide a computer-assisted proof for a new lower bound on $\mathcal{H}(4)$ in the Hilbert 16th problem, that is the maximum number of limit cycles that can occur in a polynomial planar vector field of degree 4. Indeed, we exhibit a quartic vector field for which we rigorously prove the existence of at least 24 limit cycles.

Hilbert's 16th problem is part of Hilbert's famous list of 23 problems presented in 1900 at the International Congress of Mathematicians in Paris. The second part of this problem asks whether there exists, for each natural integer n, a finite upper bound $\mathcal{H}(n)$ on the number of limit cycles a polynomial planar vector field of degree n can have [1]. For now, a proof of this conjecture seems out of reach, even for $n = 2$. A simpler version of this problem restricts the investigation to perturbed Hamiltonian systems [2]. The upper bound $\mathcal{Z}(n)$ for such systems of degree n was proved to be finite for all n. A key ingredient is the Poincaré-Pontryagin theorem that relates the number of limit cycles with the number of zeros of so-called Abelian integrals along the level curves of the potential function associated to the Hamiltonian system.

The quartic system we investigate is not Hamiltonian but still integrable, so that the Poincaré-Pontryagin theorem still applies. Our work consists in approximating the Abelian integral associated to each monomial of the perturbation (as function of the energy level of the potential function), adjusting the coefficients to maximize the number of zeros of the resulting linear combination, and finally computing rigorous enclosures of the integral at various points to certify the number of sign changes. For that purpose, we use Rigorous Polynomial Approximations [4, 3] via our free C library available here[1], which allows us to perform rigorous and efficient evaluations of the Abelian integral.

A formalization of this result in Coq is in progress, in order to avoid possible implementation errors that may lead to a wrong result.

References

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[1] The TchebyApprox experimental library : `https://gforge.inria.fr/projects/tchebyapprox/`

[2] V. I. Arnol'd, *Ten problems*, vol. 1 of Adv. Soviet Math., Amer. Math. Soc., Providence, RI, 1990.

[3] F. Bréhard, N. Brisebarre, and M. Joldeş, *Validated and numerically efficient Chebyshev spectral methods for linear ordinary differential equations*, ACM Trans. Math. Softw., 44 (2018), pp. 44:1–44:42, https://doi.org/10.1145/3208103.

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