Sparse Gröbner basis algorithms for solving polynomial systems

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Solving polynomial systems is one of the oldest and most important problems in computational mathematics and has many applications in several domains of science and engineering. It is an intrinsically hard problem with complexity at least single exponential in the number of variables [8]. However, almost always, the polynomial systems coming from applications have some kind of a structure. For example, several problems in computer-aided design, robotics, computer vision, molecular biology and kinematics [4, 5] involve polynomial systems that are sparse, that is just few monomials have non-zero coefficients. In this talk, we focus on exploiting the sparsity of the supports of the polynomials to solve the systems faster than the worst case estimates. In this setting a system is unmixed if all its polynomials have the same support (Newton polytope), and mixed otherwise. We will concentrate in sparse Gröbner basis algorithms for solving mixed systems.

Gröbner bases are particular kind of bases which allow us to compute geometric and algebraic properties of ideals and modules. They are in the heart of most of the nonlinear algebra algorithms. They are standard tools to solve 0-dimensional systems ; we do so by computing them with respect to a lexicographical monomial order. The efficient computation of Gröbner basis relies on recursive computations, avoiding reductions to zero [1]. The main criterion to avoid these reductions is the F5 criterion [6]. The term sparse Gröbner bases, as defined in [7], refers to the computation of Gröbner basis over a semigroup algebra. The idea is to embed the polynomials in a semigroup algebra, instead of the standard polynomial algebra, and so to force the computations to take into account the sparsity. In [7], the authors consider unmixed systems and introduce an algorithm to compute such a basis which, under regularity assumptions, performs no reduction to zero. However, the mixed systems does not satisfy such regularity assumptions and so, for such systems, this algorithm does not avoid every reduction to zero. To overcome this obstacle, we propose [2] an alternative definition of sparse Gröbner basis together with an algorithm to compute it. Our novelty consists in considering sparse orders that are not monomial orders. If the *mixed system* satisfies certain regularity assumptions, our algorithm avoids every reduction to zero. These assumptions are more general than the ones in [7].

We also consider mixed multihomogeneous systems, which form an important subclass of mixed sparse systems. Some of their properties are well understood, for example, the (multigraded) Castelnuovo-Mumford regularity [10, 3]. In the standard homogeneous setting, this regularity is a measure of the complexity of a module. In [2], we use the definitions and bounds on the multigraded

Castelnuovo-Mumford regularity from [10, 3] and we extended the F5 criterion [6] to introduce a dedicated recursive algorithm that computes Gröbner bases of regular 0-dimensional mixed multihomogeneous systems, avoiding every reduction to zero. Moreover, we propose a generalization of the Macaulay bound which bounds the maximal multidegree appearing in our computations [9] : If we consider polynomials of multidegrees $\boldsymbol{d}_1, \ldots, \boldsymbol{d}_{(n_1+\cdots+n_r)}$ over the multiprojective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$, this maximal degree is $\sum_i \boldsymbol{d}_i - (n_1, \ldots, n_r) + (1, \ldots, 1)$.

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